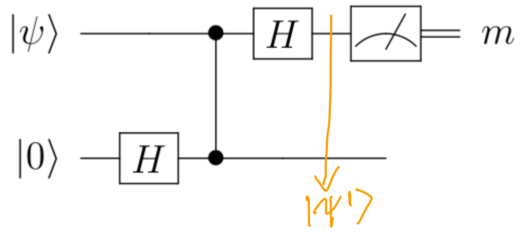
1. **Gate teleportation** (12 points: 4+2+2+1+3)

The fundamental primitive of MBQC is called gate teleportation, a simple version of which can be demonstrated in the following circuit:



(1)

Here, the entangling gate is a controlled-Z acting on a two-qubit state as $CZ\,|ab\rangle = (-1)^{ab}\,|ab\rangle$.

a) Suppose that in the above circuit we measure the first register in the Z eigenbasis. Write the resulting state on the remaining subsystem in terms of the input state, depending on the measurement outcome m (you can neglect the normalization constant)[1].

$$|\psi'\rangle = (H \otimes \mathbb{1})(C\text{-}Z)(\mathbb{1} \otimes H)\,|\psi\rangle \otimes |0\rangle =$$

$$= (H \otimes \mathbb{1})(C\text{-}Z)\,|\psi\rangle \otimes |+\rangle$$

$$= (H \otimes \mathbb{1})(C\text{-}Z)(a|0\rangle + b|1\rangle) \otimes |+\rangle$$

$|\psi\rangle = a|0\rangle + b|1\rangle$

$$= (H \otimes \mathbb{1})(a\,|0\rangle \otimes |+\rangle + b\,|1\rangle \otimes |-\rangle)$$

$C\text{-}Z\,|0\rangle \otimes |+\rangle = |0\rangle \otimes |+\rangle$

$C\text{-}Z\,|1\rangle \otimes |+\rangle = |1\rangle \otimes |-\rangle$

$$= a\,|+\rangle \otimes |+\rangle + b\,|-\rangle \otimes |-\rangle$$

- MEASUREMENT $\{|0\rangle\langle 0| \otimes \mathbb{1},\ |1\rangle\langle 1| \otimes \mathbb{1}\}$

- $|m\rangle\langle m| \otimes \mathbb{1}\,|\psi'\rangle = a\,\langle m|+\rangle\,|m\rangle \otimes |+\rangle + b\,\langle m|-\rangle\,|m\rangle \otimes |-\rangle =$

  $m = 0, 1$

$$= \frac{a}{\sqrt{2}}\,|m\rangle \otimes |+\rangle + \frac{(-1)^m}{\sqrt{2}}\,b\,|m\rangle \otimes |-\rangle$$

$$= |m\rangle \otimes \frac{1}{\sqrt{2}}\left(a|+\rangle + b(-1)^m|-\rangle\right)$$

$$= \frac{1}{\sqrt{2}} |m\rangle \otimes \left(a|+\rangle + b(X)^m|-\rangle\right)$$

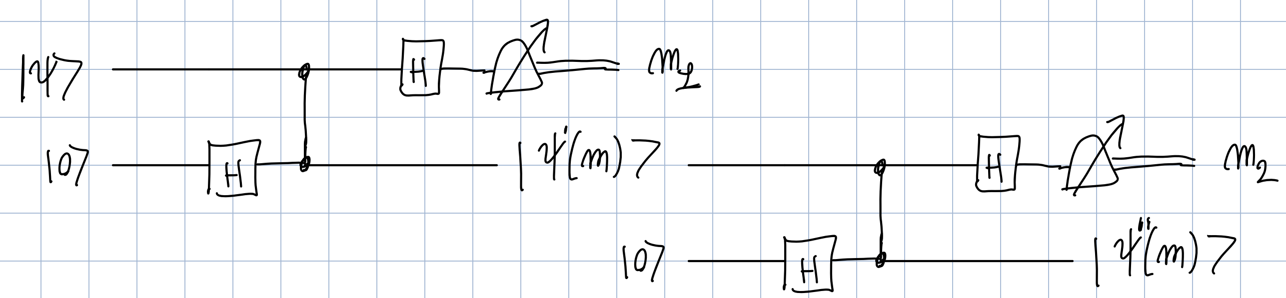
$$= \frac{1}{\sqrt{2}} |m\rangle \otimes (X)^m \left(a|+\rangle + b|-\rangle\right)$$

$X^m|+\rangle = |+\rangle$

$$= \frac{1}{\sqrt{2}} |m\rangle \otimes X^m H|\psi\rangle$$

$$\Rightarrow |\psi'(m)\rangle = X^m H|\psi\rangle$$

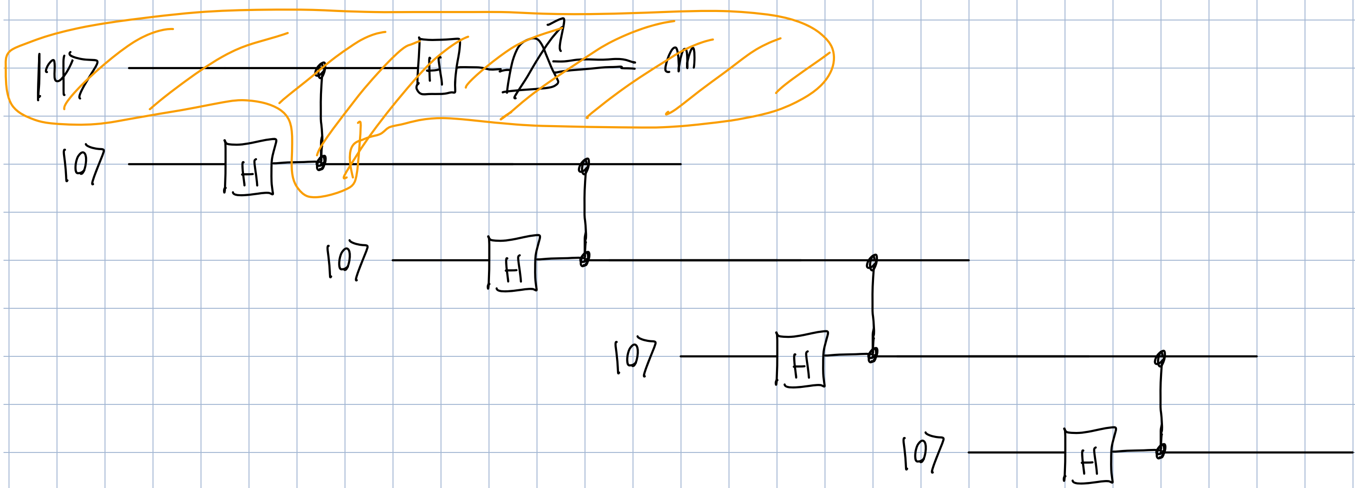
b) Imagine taking the output state of the second wire, denoted $|\psi'(m_1)\rangle$, following a measurement with outcome m_1, and feeding it back to a similar circuit, with measurement outcome m_2. Can you write the output state in terms of $|\psi\rangle$? *Hint: you shouldn't need to do any calculation.*



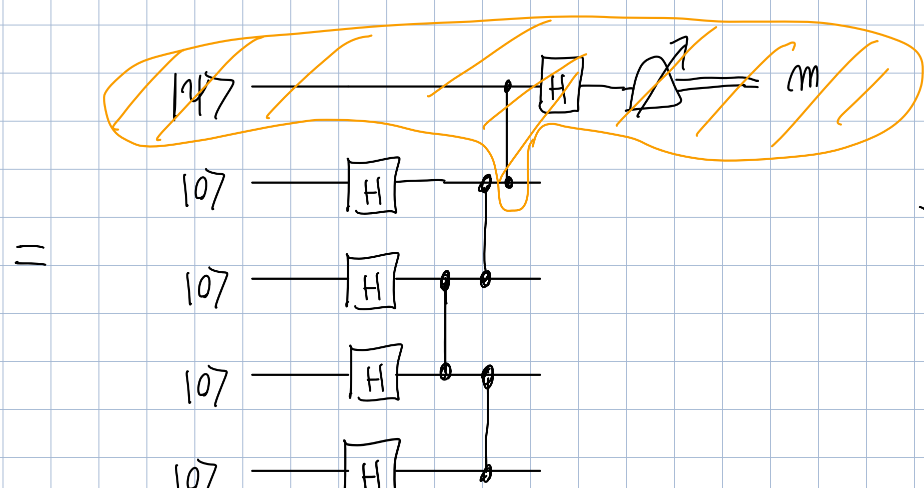
$$|\psi''(m_2)\rangle = X^{m_2} H|\psi'(m_1)\rangle = X^{m_2} H X^{m_1} H|\psi\rangle$$

A key insight in MBQC is that if we want to repeat the above process n times we can prepare an entangled n-qubit resource state $|\Gamma\rangle$ *beforehand*, independent of the input state $|\psi\rangle$. $|\Gamma\rangle$ can be depicted as a one-dimensional strip of pair-wise entangled qubits, called a 1-d *cluster state*. We can then entangle $|\psi\rangle$ to the first qubit of the strip and subsequently only perform measurements (and possibly single-qubit Pauli corrections to remove the dependency of the output on measurement outcomes). Since $\langle Z = \pm 1|\, H = \langle X = \pm 1|$, you can convince yourself that in circuit 1 after the CZ the first qubit is effectively measured in the X basis. In the following point, we consider the H gates right before the computational basis measurement as "part of an X measurement process".

c) Draw a sketch of the circuit resulting from the n-fold repetition of the citcuit in Eq. 1 and write an expression for the resource state $|\Gamma\rangle$ (*Hint: CZ gates on different qubits all commute and isolate all measurements at the end of the circuit.*)



$$C\text{-}Z_{1,2}\, C\text{-}Z_{2,3}\,|x_1, x_2, x_3\rangle = (-1)^{x_2 \cdot x_3}\, C\text{-}Z_{1,2}\,|x_1, x_2, x_3\rangle = (-1)^{x_2 \cdot x_3}(-1)^{x_1 \cdot x_2}\,|x_1, x_2, x_3\rangle = C\text{-}Z_{2,3}\, C\text{-}Z_{1,2}\,|x_1, x_2, x_3\rangle$$



$$|\Gamma\rangle = \left(\prod_{i=1}^{[n/2]} C\text{-}Z_{(2i-1),2i}\right) H^{\otimes n}\,|0\rangle^{\otimes n}$$

d) Consider $R_z(\theta) = \exp(-\mathrm{i}\frac{\theta}{2}Z)$ and define $X_\theta = R_z(\theta)^\dagger X R_z(\theta)$. Show that

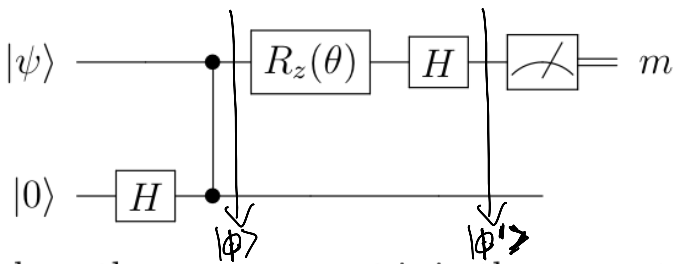
$$X_\theta R_z^\dagger(\theta)\, H\, |Z = m\rangle = (-1)^m R_z^\dagger(\theta)\, H\, |Z = m\rangle\,. \tag{2}$$

$$X_\theta R_z^\dagger(\theta) H |m\rangle = R_z^\dagger(\theta) X H |m\rangle =$$

$$\underset{H^2 = 1}{=} R_z^\dagger(\theta) H \underbrace{H X H}_{Z} |m\rangle$$

$$= (-1)^m R_z^\dagger(\theta) H |m\rangle$$

e) Consider the following circuit



(3)

where the measurement is in the computational basis. What is the observable that is effectively measured on the first qubit after the CZ? And what is the output state of the circuit, depending on the measurement outcome? *Hint: the* $R_z(\theta)$ *commutes with* CZ.

- $\langle \phi' | (Z \otimes \mathbb{1}) | \phi' \rangle = \langle \phi | (H R_z(\theta))^\dagger \otimes \mathbb{1} \, (Z \otimes \mathbb{1}) \, (H R_z(\theta)) \otimes \mathbb{1} | \phi \rangle$

$$= \langle \phi | (R_z^\dagger(\theta) H \otimes \mathbb{1}) (Z \otimes \mathbb{1}) (H R_z(\theta) \otimes \mathbb{1}) | \phi \rangle$$

$\Rightarrow$ The observable is $R_z^\dagger(\theta) \underbrace{H Z H}_{= X} R_z(\theta) = R_z^\dagger X R_z(\theta) = X_\theta$

$$\left(\begin{aligned} C-Z &= |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes Z \\ &\overset{\pi}{=} e^{i\pi \left(\frac{\mathbb{1}-Z_1}{2}\right)\left(\frac{\mathbb{1}-Z_2}{2}\right)} \end{aligned} \right)$$

they act on $|x_1, x_2\rangle$ equally.

- $|m\rangle\langle m| \otimes \mathbb{1} \, |\phi'\rangle = \left(|m\rangle\langle m| \otimes \mathbb{1}\right)\left(H R_z(\theta) \otimes \mathbb{1}\right)|\phi\rangle$

$$= \left(|m\rangle\langle m| \otimes \mathbb{1}\right)\left(H R_z(\theta) \otimes \mathbb{1}\right) C-Z \left(\mathbb{1} \otimes H\right)|\psi\rangle \otimes |0\rangle$$

$$= \left(|m\rangle\langle m| \otimes \mathbb{1}\right)\left(H \otimes \mathbb{1}\right) C-Z \left(\mathbb{1} \otimes H\right)\left(R_z(\theta) \otimes \mathbb{1}\right)|\psi\rangle \otimes |0\rangle$$

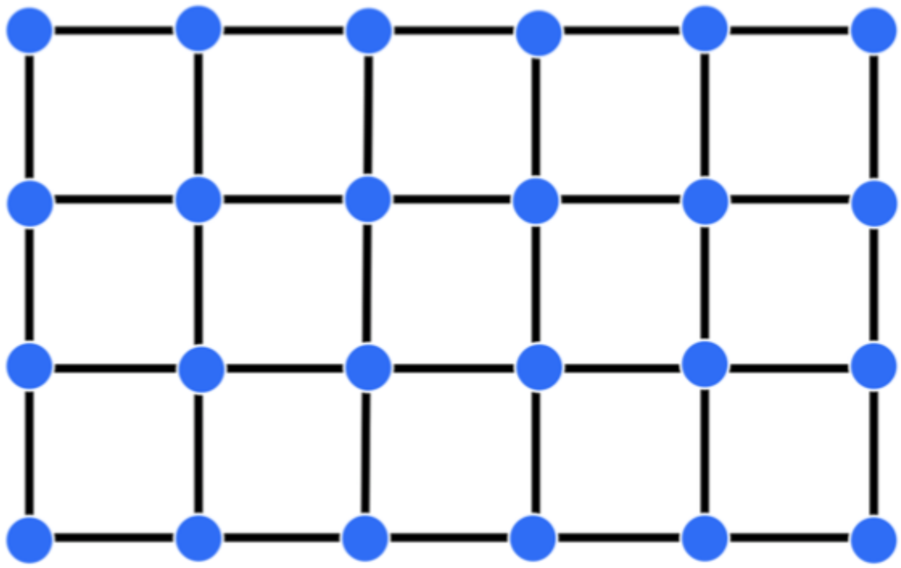
$$= \underbrace{\left(|m\rangle\langle m| \otimes \mathbb{1}\right)\left(H \otimes \mathbb{1}\right) C-Z \left(\mathbb{1} \otimes H\right)}_{\text{We had it before}} R_z(\theta)|\psi\rangle \otimes |0\rangle$$

$$\Rightarrow \quad |\psi'(m)\rangle = X^m H \, R_z(\theta)|\psi\rangle$$

Since we get the Hadamard gate "for free" according to circuit 1, this result, together with the results in previous sheets shows us that a 1-d cluster state and single-qubit measurements are sufficient to perform an arbitrary single-qubit operation. The result is furthermore deterministic if we can operate corrective X operations depending on measurement outcomes.

2. **Universal Quantum Computation with Cluster States** 8 Points: 3 + 1 + 2 + 2

In this exercise we consider a two dimensional cluster state where qubits are arranged in a rectangular grid



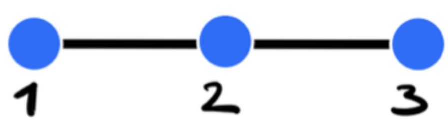
where nodes are qubit registers. This state can be obtained by preparing each qubit in the $|+\rangle$ state and applying CZ operations between qubits connected by an edge. This general procedure can be used to produce states represented by any graph, which are simply called *graph states*. Cluster states, described by a rectangular grid, lie at the core of measurement based quantum computation (MBQC) because, given one such state, one can perform *any* quantum computation with single qubit measurements (in various bases), provided the rectangular patch is large enough. In the present exercise, we will sketch a proof of this fact. According to the previous exercise, it is sufficient to show that we can perform two-qubit gates. To this end, let us start with some definitions.

The stabilizer formalism is once again useful to compactly describe what is going on. The stabilizer generators of an arbitrary graph state $|\Gamma\rangle$, with Γ some graph, are given by

$$\mathcal{S} = \{X_a \prod_{i \sim a} Z_i \mid a \in \Gamma\}, \tag{4}$$

where $i \sim a$ denotes the set of qubits adjacent (connected) to qubit a. In paticular, it holds that $S_a |\Gamma\rangle = |\Gamma\rangle \ \forall S_a \in \mathcal{S}$.

a) Consider the graph state represented by



Write down the stabilizer generators of this state according to Eq. 4 and check that they indeed stabilize the state. *Hint: write the stabilizer of the state by conjugating the ones of the $|+++\rangle$ state by the appropriate CZs.*

- The state is $CZ_{(1,2)} CZ_{(2,3)} |+\rangle^{\otimes 3}$
- stabilizers for $|+\rangle^{\otimes 3}$ are $\langle X_1, X_2, X_3 \rangle$
- stabilizers for $CZ_{(2,3)} |+\rangle^{\otimes 3}$ are:

$$\langle CZ_{2,3} X_1 CZ_{2,3}^\dagger,\ CZ_{2,3} X_2 (CZ_{2,3})^\dagger,\ CZ_{2,3} X_3 (CZ_{2,3})^\dagger \rangle$$

$$= \langle X_1,\ X_2 Z_3,\ Z_2 X_3 \rangle$$

$\uparrow$

$$CZ_{2,3} X_2 (CZ_{2,3})^\dagger |x_2, x_3\rangle = CZ_{2,3} X_2 (-1)^{x_2 \cdot x_3} |x_2, x_3\rangle$$
$$= (-1)^{x_2 \cdot x_3 + \bar{x}_2 \cdot x_3} |\bar{x}_2, x_3\rangle$$
$$= (-1)^{x_3} |\bar{x}_2, x_3\rangle =$$
$$= X_2 Z_3 |x_2, x_3\rangle$$

- $CZ_{2,3} X_3 CZ_{2,3} |x_2, x_3\rangle = (-1)^{x_2 \cdot x_3 + x_2 \cdot \bar{x}_3} |x_2, \bar{x}_3\rangle$

$$= (-1)^{x_2} |x_2, \bar{x}_3\rangle$$
$$= Z_2 X_3$$

- stabilizers for $CZ_{(1,2)} CZ_{(2,3)} |+\rangle^{\otimes 3}$:

$$CZ_{(1,2)} \langle X_1,\ X_2 Z_3,\ Z_2 X_3 \rangle CZ_{(1,2)} =$$

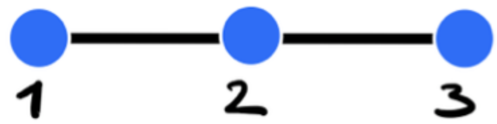
$$= \langle X_1 Z_2,\ Z_1 X_2 Z_3,\ Z_2 X_3 \rangle$$

$\uparrow$

- $CZ_{1,2} X_1 CZ_{1,2} |x_1, x_2\rangle = (-1)^{x_1 \cdot x_2 + \bar{x}_1 \cdot x_2} |\bar{x}_1, x_2\rangle = X_1 Z_2 |x_1, x_2\rangle$

We now use stabilizers to prove two useful tricks that allow us to easily modify the shape of a 2-d cluster state. Remember that when we have a stabilizer state with stabilizer generators $\{S_j\}$ and we want to measure some Pauli operator O, we can represent the action of a measurement as follows: if O commutes with all stabilizers, the measurement result is predetermined and the state is unchanged (being already an eigenstate of O. If O does not commute with some stabilizers, we can find a set of generators such that only one generator, say S_1, anti-commutes with O. This set can be found by multiplying some of the generators together. Following the measurement, we replace S_1 with $(-1)^m O$, where m is the measurement outcome.

b) Consider again the graph state represented by



Show that measuring the second qubit in the X basis the stabilizers of the post-measurement state are those of a two-qubit cluster, apart for a Hadamard on either the first or the third qubit and measurement-dependent phases. *Hint: start by finding a set of generators such that only one anti-commutes with the measurement. Multiply then the post-measurement stabilizers to remove unwanted dependencies on operators acting on the measured qubit.*

- The stabilizers are:

$$\langle \underbrace{X_1 Z_2}_{g_1},\ \underbrace{Z_1 X_2 Z_3}_{g_2},\ \underbrace{Z_2 X_3}_{g_3} \rangle$$

- X_2 commutes with $Z_1 X_2 Z_3$, but does not commute with $X_1 Z_2$, $Z_2 X_3$.

$$\Rightarrow \langle X_1 Z_2,\ Z_1 X_2 Z_3,\ g_1 g_3 \rangle, \quad g_1 g_3 = X_1 X_3$$

- X_2 commutes with all apart from $X_1 Z_2$.

- After measurement, the stab. generators will be:

$$\langle (-1)^m X_2,\ Z_1 X_2 Z_3,\ X_1 X_3 \rangle = \langle (-1)^m X_2,\ (-1)^m Z_1 Z_3,\ X_1 X_3 \rangle =$$

$= \langle (-1)^m X_2, (-1)^m Z_1 Z_3, X_1 X_3 \rangle$

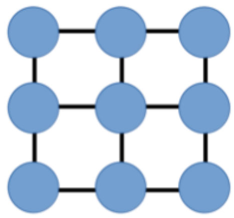
- Apply Hadamard on the first qubit $\Rightarrow$

$$\langle (-1)^m X_2, (-1)^m (H Z_1 H) Z_3, (H X_1 H) X_3 \rangle$$

$$= \langle (-1)^m X_2, (-1)^m X_1 Z_3, Z_1 X_3 \rangle$$

The above result shows that we can "shorten" wires to connect initially distant qubits on the lattice. The second equivalence is obtained multiplying $Z_1 X_2 Z_3$ by $\pm X_2$.

c) Consider now the 3×3 square cluster state



Show that measuring Z on the central node effectively disentangles it from the rest of the state, leaving the other qubits in a graph state.

- The stabiliser generators are:

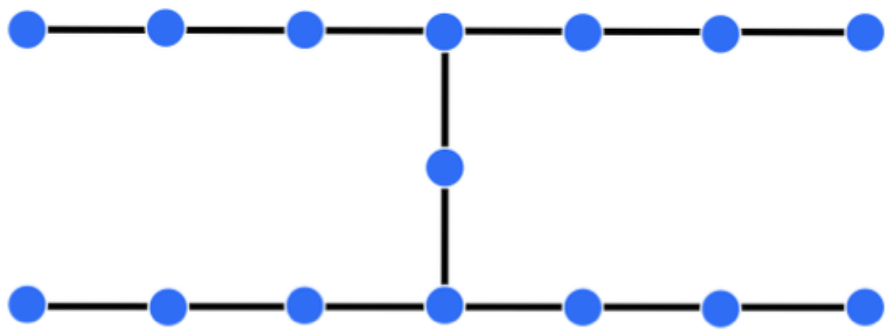
$$\mathcal{S} = \{X_a \prod_{i \sim a} Z_i \mid a \in \Gamma\},$$

- They all commute with Z_{CENTER} apart from the stabilizer generator with X_{CENTER}.
- After measurements "m", the stabilizer with X_{CENTER} is substitute with $(-1)^m \cdot Z_{\text{CENTER}}$, while the others remain the same apart from a phase $(-1)^m$ instead of Z_{CENTER}.

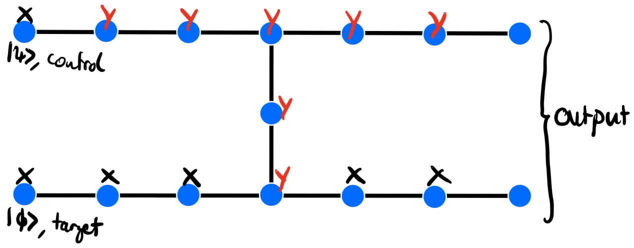
These two tricks can be generalized to show that, given a 2D cluster state, one can "cut out" any 2D regular grid and obtain the graph state needed to implement some circuit by single qubit measurements on appropriate sites. This justifies using the graph shape in the following point.

Finally, we turn to the $CNOT$ gate. We can apply the $CNOT$ gate in the MBQC scheme by using the following graph state.

d) Consider the following graph state:

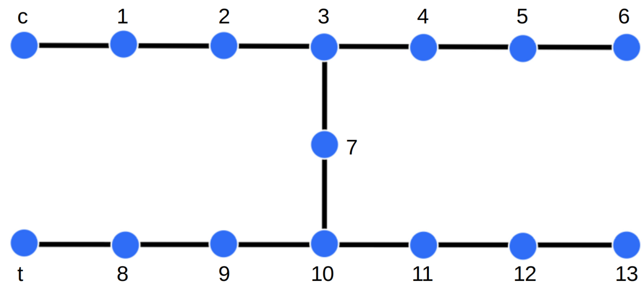


Show that the following measurements implement a CNOT gate between the twoinput states $|\psi\rangle$ and $|\phi\rangle$ up to local pauli corrections:



Hint: There are two ways to prove this. Either, one explicitly calculates the output of the full circuit corresponding to the preparation and the measurements or one uses the stabilizer formalism where one only has to keep track how the stabilizers of the graph state change during the measurements. You might also have a look at https://arxiv.org/pdf/quant-ph/0301052.pdf.

Solution: Label the top qubits in the graph state excluding the input $1, 2, \ldots, 6$ from left to right, the central one as 7 and the bottom ones $8, \ldots, 13$, again left to right. Label c and t the input control and target, respectively, qubits 6 and 13 are the corresponding output qubits:



To show that the pattern of measurements applies a $CNOT$ it is sufficient to specify its action on a basis of the 2-qubit inputs. Note that computational basis states $|jk\rangle$ on the inputs are stabilized by $\left\{(-1)^j Z_c, (-1)^k Z_t\right\}$. One can see that

$$CNOT\left((-1)^j Z_c, (-1)^k Z_t\right) CNOT = \left((-1)^j Z_c, (-1)^k Z_c Z_t\right). \tag{12}$$

We will try and show that after the measurements the stabilizers of the output qubits are equal to these modulo single-qubit Pauli rotations.

From the previous points, the stabilizers of the graph state on all other qubits (before coupling with the input) are generated by

$$\left\{\begin{array}{c} X_1Z_2,\ X_2Z_1Z_3,\ X_3Z_2Z_4Z_7, \quad X_4Z_3Z_5,\ X_5Z_4Z_6,\ X_6Z_5, \\ X_7Z_3Z_{10} \\ X_8Z_9,\ X_9Z_8Z_{10},\ X_{10}Z_7Z_9Z_{11}, \quad X_{11}Z_{10}Z_{12},\ X_{12}Z_{11}Z_{13},\ X_{13}Z_{12} \end{array}\right\}. \tag{13}$$

After we apply the two CZ to couple in the input, the stabilizer generators (SG) of the overall state become

$$\left\{\begin{array}{c} (-1)^j Z_c,\ Z_cX_1Z_2,\ X_2Z_1Z_3,\ X_3Z_2Z_4Z_7, \quad X_4Z_3Z_5,\ X_5Z_4Z_6,\ X_6Z_5, \\ X_7Z_3Z_{10} \\ (-1)^k Z_t,\ Z_tX_8Z_9,\ X_9Z_8Z_{10},\ X_{10}Z_7Z_9Z_{11}, \quad X_{11}Z_{10}Z_{12},\ X_{12}Z_{11}Z_{13},\ X_{13}Z_{12} \end{array}\right\}. \tag{14}$$

For each measurement in the following we first redefine SG such that only one anti-commutes with the measured operator and then we replace the anti-commuting operator with $\pm O$ where O is the measured observable.

We start by considering measurement of X_c, X_t. We multiply $Z_c X_1 Z_2$ by $(-1)^j Z_c$ and $Z_t X_8 Z_9$ by $(-1)^k Z_t$. Now only $(-1)^j Z_c$ anti-commutes with X_c and only $(-1)^k Z_t$ anti-commutes with X_t. Following the measurement, with outcomes m_c, m_t, we have SG

$$\left\{ \begin{array}{cc} (-1)^{m_c} X_c,\ (-1)^j X_1 Z_2,\ X_2 Z_1 Z_3,\ X_3 Z_2 Z_4 Z_7, & X_4 Z_3 Z_5,\ X_5 Z_4 Z_6,\ X_6 Z_5, \\ X_7 Z_3 Z_{10} & \\ (-1)^{m_t} X_t,\ (-1)^k X_8 Z_9,\ X_9 Z_8 Z_{10},\ X_{10} Z_7 Z_9 Z_{11}, & X_{11} Z_{10} Z_{12},\ X_{12} Z_{11} Z_{13},\ X_{13} Z_{12} \end{array} \right\}. \tag{15}$$

Note that X_c and X_t commute with all the remaining measurements, so they will no longer change in the following.

We now measure Y_1, X_8, with outcomes m_1, m_8. Repeating the above procedure we have SG

$$\left\{ \begin{array}{cc} (-1)^{m_c} X_c,\ (-1)^{m_1} Y_1,\ (-1)^j Y_1 Y_2 Z_3,\ X_3 Z_2 Z_4 Z_7, & X_4 Z_3 Z_5,\ X_5 Z_4 Z_6,\ X_6 Z_5, \\ X_7 Z_3 Z_{10} & \\ (-1)^{m_t} X_t,\ (-1)^{m_8} Y_8,\ (-1)^k Y_8 Y_9 Z_{10},\ X_{10} Z_7 Z_9 Z_{11}, & X_{11} Z_{10} Z_{12},\ X_{12} Z_{11} Z_{13},\ X_{13} Z_{12} \end{array} \right\}. \tag{16}$$

We can remove the dependency on Y_1 and Y_8 for the folllowing by multiplying the appropriate stabilizers by$(-1)^{m_1} Y_1$ or $(-1)^{m_8} Y_8$ obtaining

$$\left\{ \begin{array}{cc} (-1)^{m_c} X_c,\ (-1)^{m_1} Y_1,\ (-1)^j (-1)^{m_1} Y_2 Z_3,\ X_3 Z_2 Z_4 Z_7, & X_4 Z_3 Z_5,\ X_5 Z_4 Z_6,\ X_6 Z_5, \\ X_7 Z_3 Z_{10} & \\ (-1)^{m_t} X_t,\ (-1)^{m_8} Y_8,\ (-1)^k (-1)^{m_8} Y_9 Z_{10},\ X_{10} Z_7 Z_9 Z_{11}, & X_{11} Z_{10} Z_{12},\ X_{12} Z_{11} Z_{13},\ X_{13} Z_{12} \end{array} \right\}, \tag{17}$$

or, ditching the stabilizers that are no longer relevant

$$\left\{ \begin{array}{cc} (-1)^{j+m_1} Y_2 Z_3,\ X_3 Z_2 Z_4 Z_7, & X_4 Z_3 Z_5,\ X_5 Z_4 Z_6,\ X_6 Z_5, \\ X_7 Z_3 Z_{10} & \\ (-1)^{k+m_8} Y_9 Z_{10},\ X_{10} Z_7 Z_9 Z_{11}, & X_{11} Z_{10} Z_{12},\ X_{12} Z_{11} Z_{13},\ X_{13} Z_{12} \end{array} \right\}, \tag{18}$$

Measure now Y_2 and X_9. We get

$$\left\{ \begin{array}{cc} (-1)^{j+m_1} Y_2 Z_3,\ (-1)^{m_2} Y_2, & X_4 Z_3 Z_5,\ X_5 Z_4 Z_6,\ X_6 Z_5, \\ X_7 Z_3 Z_{10} & \\ (-1)^{m_9} X_9,\ (-1)^{k+m_8+1} Z_7 X_9 Y_{10} Z_{11}, & X_{11} Z_{10} Z_{12},\ X_{12} Z_{11} Z_{13},\ X_{13} Z_{12} \end{array} \right\}, \tag{19}$$

or

$$\left\{ \begin{array}{cc} (-1)^{j+m_1+m_2} Z_3, & X_4 Z_3 Z_5,\ X_5 Z_4 Z_6,\ X_6 Z_5, \\ X_7 Z_3 Z_{10} & \\ (-1)^{k+m_8+m_9+1} Z_7 Y_{10} Z_{11}, & X_{11} Z_{10} Z_{12},\ X_{12} Z_{11} Z_{13},\ X_{13} Z_{12} \end{array} \right\} \tag{20}$$

or

$$\left\{ \begin{array}{cc} (-1)^{j+m_1+m_2} Z_3, & X_4 Z_3 Z_5,\ X_5 Z_4 Z_6,\ X_6 Z_5, \\ X_7 Z_3 Z_{10} & \\ (-1)^{k+m_8+m_9} Z_3 Y_7 X_{10} Z_{11}, & X_{11} Z_{10} Z_{12},\ X_{12} Z_{11} Z_{13},\ X_{13} Z_{12} \end{array} \right\} \tag{21}$$

where we see that the bottom row picks up a dependency on the top one. Measure now Y_7:

$$\left\{\begin{array}{c}(-1)^{j+m_1+m_2}Z_3, \quad X_4Z_3Z_5,\ X_5Z_4Z_6,\ X_6Z_5, \\ (-1)^{m_7}Y_7 \\ (-1)^{k+m_8+m_9}Z_3Y_7X_{10}Z_{11}, \quad X_{11}Z_{10}Z_{12},\ X_{12}Z_{11}Z_{13},\ X_{13}Z_{12}\end{array}\right\}, \tag{22}$$

or

$$\left\{\begin{array}{c}(-1)^{j+m_1+m_2}Z_3, \quad X_4Z_3Z_5,\ X_5Z_4Z_6,\ X_6Z_5, \\ (-1)^{k+m_7+m_8+m_9}Z_3X_{10}Z_{11}, \quad X_{11}Z_{10}Z_{12},\ X_{12}Z_{11}Z_{13},\ X_{13}Z_{12}\end{array}\right\}, \tag{23}$$

or

$$\left\{\begin{array}{c}(-1)^{j+m_1+m_2}Z_3, \quad (-1)^{j+m_1+m_2}X_4Z_5,\ X_5Z_4Z_6,\ X_6Z_5, \\ (-1)^{k+j+m_1+m_2+m_7+m_8+m_9}X_{10}Z_{11}, \quad X_{11}Z_{10}Z_{12},\ X_{12}Z_{11}Z_{13},\ X_{13}Z_{12}\end{array}\right\}, \tag{24}$$

or

$$\left\{\begin{array}{c}(-1)^{j+m_1+m_2}Z_3, \quad (-1)^{j+m_1+m_2}X_4Z_5,\ X_5Z_4Z_6,\ X_6Z_5, \\ (-1)^{k+j+m_1+m_2+m_7+m_8+m_9}X_{10}Z_{11}, \quad (-1)^{k+j+m_1+m_2+m_7+m_8+m_9}Y_{10}Y_{11}Z_{12},\ X_{12}Z_{11}Z_{13},\ X_{13}Z_{12}\end{array}\right\}. \tag{25}$$

Now we measure Y_3 and Y_{10} to get SG

$$\left\{\begin{array}{c}(-1)^{m_3}Y_3,\ (-1)^{j+m_1+m_2}X_4Z_5,\ X_5Z_4Z_6,\ X_6Z_5, \\ (-1)^{m_{10}}Y_{10},\ (-1)^{k+j+m_1+m_2+m_7+m_8+m_9}Y_{10}Y_{11}Z_{12},\ X_{12}Z_{11}Z_{13},\ X_{13}Z_{12}\end{array}\right\}, \tag{26}$$

or

$$\left\{\begin{array}{c}(-1)^{j+m_1+m_2}X_4Z_5,\ X_5Z_4Z_6,\ X_6Z_5, \\ (-1)^{k+j+m_1+m_2+m_7+m_8+m_9+m_{10}}Y_{11}Z_{12},\ X_{12}Z_{11}Z_{13},\ X_{13}Z_{12}\end{array}\right\}, \tag{27}$$

or

$$\left\{\begin{array}{c}(-1)^{j+m_1+m_2}X_4Z_5,\ (-1)^{j+m_1+m_2}Y_4Y_5Z_6,\ X_6Z_5, \\ (-1)^{k+j+m_1+m_2+m_7+m_8+m_9+m_{10}}Y_{11}Z_{12},\ (-1)^{k+j+m_1+m_2+m_7+m_8+m_9+m_{10}+1}X_{11}Y_{12}Z_{13},\ X_{13}Z_{12}\end{array}\right\}. \tag{28}$$

Now measure Y_4 and X_{11} to get

$$\left\{\begin{array}{c}(-1)^{m_4}Y_4,\ (-1)^{j+m_1+m_2}Y_4Y_5Z_6,\ X_6Z_5, \\ (-1)^{m_{11}}X_{11},\ (-1)^{k+j+m_1+m_2+m_7+m_8+m_9+m_{10}+1}X_{11}Y_{12}Z_{13},\ X_{13}Z_{12}\end{array}\right\}, \tag{29}$$

or

$$\left\{\begin{array}{c}(-1)^{j+m_1+m_2+m_4}Y_5Z_6,\ X_6Z_5, \\ (-1)^{k+j+m_1+m_2+m_7+m_8+m_9+m_{10}+m_{11}+1}Y_{12}Z_{13},\ X_{13}Z_{12}\end{array}\right\}, \tag{30}$$

or

$$\left\{ \begin{array}{c} (-1)^{j+m_1+m_2+m_4} Y_5 Z_6,\ \ X_6 Z_5, \\ (-1)^{k+j+m_1+m_2+m_7+m_8+m_9+m_{10}+m_{11}} X_{12} Y_{13},\ \ X_{13} Z_{12} \end{array} \right\}, \tag{31}$$

Finally, we need to measure Y_5 and X_{12}, leading to

$$\left\{ \begin{array}{c} (-1)^{j+m_1+m_2+m_4} Y_5 Z_6,\ (-1)^{m_5} Y_5, \\ (-1)^{k+j+m_1+m_2+m_7+m_8+m_9+m_{10}+m_{11}+1} Y_{12} Z_{13},\ (-1)^{m_{12}} X_{12} \end{array} \right\} \tag{32}$$

or

$$\left\{ \begin{array}{c} (-1)^{j+m_1+m_2+m_4+m_5} Z_6, \\ (-1)^{k+j+m_1+m_2+m_7+m_8+m_9+m_{10}+m_{11}+m_{12}} Z_{13} \end{array} \right\}, \tag{33}$$

or

$$\left\{ \begin{array}{c} (-1)^{m_1+m_2+m_4+m_5} \left[(-1)^j Z_6\right], \\ (-1)^{m_4+m_5+m_7+m_8+m_9+m_{10}+m_{11}+m_{12}} \left[(-1)^k Z_6 Z_{13}\right] \end{array} \right\}. \tag{34}$$

Remember now that $XZX = -XXZ = -Z$ so we can remove the measurement-dependent phases by applying single-qubit Pauli gates, namely $X^{m_1+m_2+m_4+m_5}$ to qubit 6 and $X^{m_4+m_5+m_7+m_8+m_9+m_{10}+m_{11}+m_{12}}$ to qubit 13.

After these corrective operations qubit 6 and 13 will be in the state whose SG read

$$\left\{(-1)^j Z_6,\ (-1)^k Z_6 Z_{13}\right\} \tag{35}$$

which, according to Eq. 12 means their state is $CNOT \left|j\right\rangle_6 \left|k\right\rangle_{13}$. This completes the proof.

Freie Universität Berlin

Tutorials on Quantum Information Theory

Winter term 2022/23

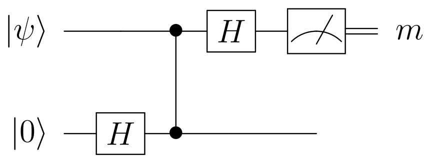
Problem Sheet 11

Measurement based quantum computing

J. Eisert, A. Townsend-Teague, A. Mele, A. Burchards, J. Denzler

1. **Gate teleportation** (12 points: 4+2+2+1+3)

The fundamental primitive of MBQC is called gate teleportation, a simple version of which can be demonstrated in the following circuit:



(1)

Here, the entangling gate is a controlled-Z acting on a two-qubit state as $CZ\,|ab\rangle = (-1)^{ab}\,|ab\rangle$.

a) Suppose that in the above circuit we measure the first register in the Z eigenbasis. Write the resulting state on the remaining subsystem in terms of the input state, depending on the measurement outcome m (you can neglect the normalization constant)[1].

b) Imagine taking the output state of the second wire, denoted $|\psi'(m_1)\rangle$, following a measurement with outcome m_1, and feeding it back to a similar circuit, with measurement outcome m_2. Can you write the output state in terms of $|\psi\rangle$? *Hint: you shouldn't need to do any calculation.*

A key insight in MBQC is that if we want to repeat the above process n times we can prepare an entangled n-qubit resource state $|\Gamma\rangle$ *beforehand*, independent of the input state $|\psi\rangle$. $|\Gamma\rangle$ can be depicted as a one-dimensional strip of pair-wise entangled qubits, called a 1-d *cluster state*. We can then entangle $|\psi\rangle$ to the first qubit of the strip and subsequently only perform measurements (and possibly single-qubit Pauli corrections to remove the dependency of the output on measurement outcomes). Since $\langle Z = \pm 1|\, H = \langle X \pm 1|$, you can convince yourself that in circuit 1 after the CZ the first qubit is effectively measured in the X basis. In the following point, we consider the H gates right before the computational basis measurement as "part of an X measurement process".

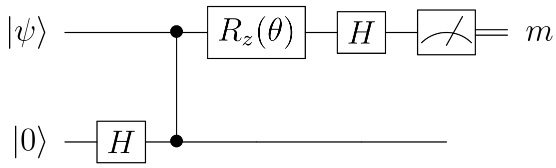
c) Draw a sketch of the circuit resulting from the n-fold repetition of the citcuit in Eq. 1 and write an expression for the resource state $|\Gamma\rangle$ (*Hint: CZ gates on different qubits all commute and isolate all measurements at the end of the circuit.*)

d) Consider $R_z(\theta) = \exp(-\mathrm{i}\frac{\theta}{2}Z)$ and define $X_\theta = R_z(\theta)^\dagger X R_z(\theta)$. Show that

$$X_\theta R_z^\dagger(\theta)\, H\, |Z = m\rangle = (-1)^m R_z^\dagger(\theta)\, H\, |Z = m\rangle\,. \qquad (2)$$

[1] In the context of MBQC, the measurements are often assumed to be "destructive", in the sense that the measured qubits are consumed by the measurement process and therefore not included in the description of what happens next. This reflects the physical reality where qubits might for example be encoded in travelling photons which are absorbed by a detector during the measurement.

e) Consider the following circuit



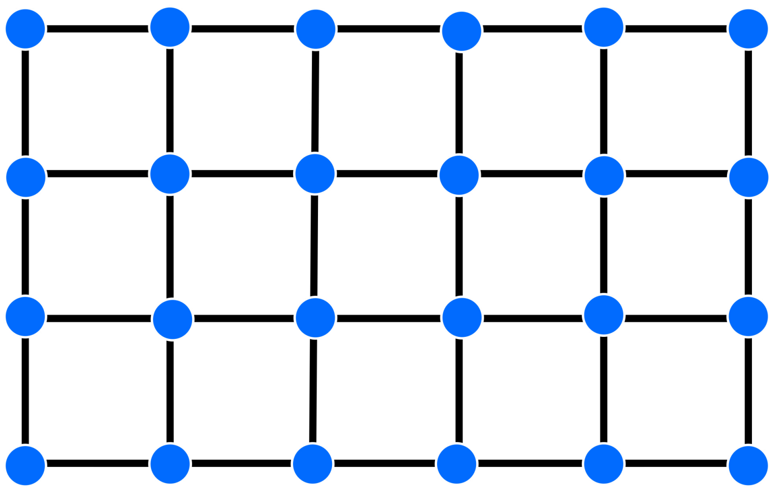
(3)

where the measurement is in the computational basis. What is the observable that is effectively measured on the first qubit after the CZ? And what is the output state of the circuit, depending on the measurement outcome? *Hint: the $R_z(\theta)$ commutes with CZ.*

Since we get the Hadamard gate "for free" according to circuit 1, this result, together with the results in previous sheets shows us that a 1-d cluster state and single-qubit measurements are sufficient to perform an arbitrary single-qubit operation. The result is furthermore deterministic if we can operate corrective X operations depending on measurement outcomes.

2. **Universal Quantum Computation with Cluster States** 8 Points: 3 + 1 + 2 + 2

In this exercise we consider a two dimensional cluster state where qubits are arranged in a rectangular grid



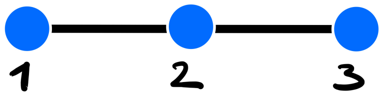
where nodes are qubit registers. This state can be obtained by preparing each qubit in the $|+\rangle$ state and applying CZ operations between qubits connected by an edge. This general procedure can be used to produce states represented by any graph, which are simply called *graph states*. Cluster states, described by a rectangular grid, lie at the core of measurement based quantum computation (MBQC) because, given one such state, one can perform *any* quantum computation with single qubit measurements (in various bases), provided the rectangular patch is large enough. In the present exercise, we will sketch a proof of this fact. According to the previous exercise, it is sufficient to show that we can perform two-qubit gates. To this end, let us start with some definitions.

The stabilizer formalism is once again useful to compactly describe what is going on. The stabilizer generators of an arbitrary graph state $|\Gamma\rangle$, with Γ some graph, are given by

$$\mathcal{S} = \{X_a \prod_{i\sim a} Z_i \mid a \in \Gamma\}, \tag{4}$$

where $i \sim a$ denotes the set of qubits adjacent (connected) to qubit a. In paticular, it holds that $S_a |\Gamma\rangle = |\Gamma\rangle \ \forall S_a \in \mathcal{S}$.

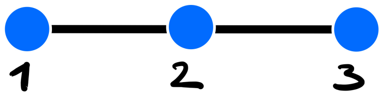
a) Consider the graph state represented by



Write down the stabilizer generators of this state according to Eq. 4 and check that they indeed stabilize the state. *Hint: write the stabilizer of the state by conjugating the ones of the* $|+++\rangle$ *state by the appropriate* CZ*s.*

We now use stabilizers to prove two useful tricks that allow us to easily modify the shape of a 2-d cluster state. Remember that when we have a stabilizer state with stabilizer generators $\{S_j\}$ and we want to measure some Pauli operator O, we can represent the action of a measurement as follows: if O commutes with all stabilizers, the measurement result is predetermined and the state is unchanged (being already an eigenstate of O. If O does not commute with some stabilizers, we can find a set of generators such that only one generator, say S_1, anti-commutes with O. This set can be found by multiplying some of the generators together. Following the measurement, we replace S_1 with $(-1)^m O$, where m is the measurement outcome.

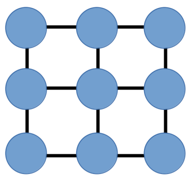
b) Consider again the graph state represented by



Show that measuring the second qubit in the X basis the stabilizers of the post-measurement state are those of a two-qubit cluster, apart for a Hadamard on either the first or the third qubit and measurement-dependent phases. *Hint: start by finding a set of generators such that only one anti-commutes with the measurement. Multiply then the post-measurement stabilizers to remove unwanted dependencies on operators acting on the measured qubit.*

The above result shows that we can "shorten" wires to connect initially distant qubits on the lattice. The second equivalence is obtained multiplying $Z_1X_2Z_3$ by $\pm X_2$.

c) Consider now the 3×3 square cluster state

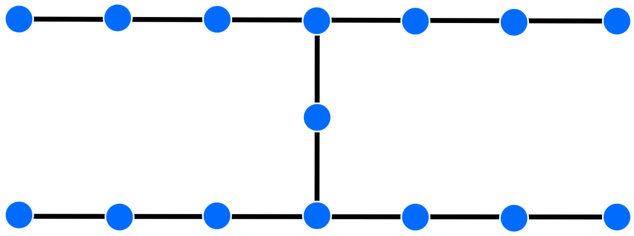


Show that measuring Z on the central node effectively disentangles it from the rest of the state, leaving the other qubits in a graph state.

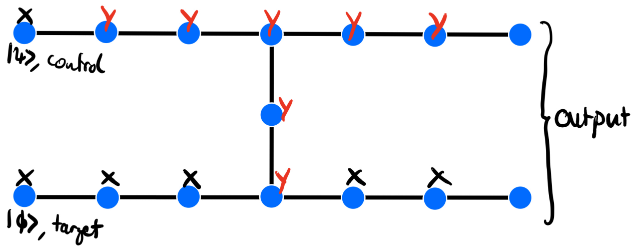
These two tricks can be generalized to show that, given a 2D cluster state, one can "cut out" any 2D regular grid and obtain the graph state needed to implement some circuit by single qubit measurements on appropriate sites. This justifies using the graph shape in the following point.

Finally, we turn to the $CNOT$ gate. We can apply the $CNOT$ gate in the MBQC scheme by using the following graph state.

d) Consider the following graph state:



Show that the following measurements implement a CNOT gate between the twoinput states $|\psi\rangle$ and $|\phi\rangle$ up to local pauli corrections:



Hint: There are two ways to prove this. Either, one explicitly calculates the output of the full circuit corresponding to the preparation and the measurements or one uses the stabilizer formalism where one only has to keep track how the stabilizers of the graph state change during the measurements. You might also have a look at https://arxiv.org/pdf/quant-ph/0301052.pdf.