1. **Operator Properties**

In this exercise we take a short break from following the main content covered in the lecture and return back to proving some simple but useful identities for operators on complex Hilbert spaces. In particular, we explore the two important facts that operators are completely specified by their diagonal elements in all bases as well as the power of the square root representation for positive operators.

a) Interestingly in a complex inner product space an operator is fully specified when its diagonal elements in all bases are known. Show this by verifying the identity

$$\langle \phi | A | \psi \rangle = \frac{1}{4} \sum_{k=0}^{3} i^k \langle \psi + i^k \phi | A | \psi + i^k \phi \rangle . \quad (1)$$

- $\sum_{k=0}^{3} i^k \langle \psi + i^k \phi | A | \psi + i^k \phi \rangle = \underbrace{\sum_{k=0}^{3} i^k}_{=0} \langle \psi | A | \psi \rangle + \underbrace{\sum_{k=0}^{3} i^k (i)^k}_{= \sum_k (-1)^k = 0} \langle \psi | A | \phi \rangle$

$\underbrace{\sum_{k=0}^{3} i^k (-i)^k}_{= \sum_k (1)^k = 4} \langle \phi | A | \psi \rangle + \underbrace{\sum_{k=0}^{3} (i)^{2k} (-i)^k}_{= \sum_k (i)^k = 0} \langle \phi | A | \phi \rangle =$

$= 4 \langle \phi | A | \psi \rangle$

b) Use the previous identity to show that $\forall \psi : \langle \psi | A | \psi \rangle = \langle \psi | B | \psi \rangle \implies A = B$.

- $A = B \iff \langle V_1 | A | V_2 \rangle = \langle V_1 | B | V_2 \rangle \quad \forall V_1, V_2 .$

- $\langle V_1 | A | V_2 \rangle = \frac{1}{4} \sum_{k=0}^{3} i^k \underbrace{\langle V_2 + i^k V_1 | A | V_2 + i^k V_1 \rangle}_{= \langle V_2 + i^k V_1 | B | V_2 + i^k V_1 \rangle} =$

$= \langle V_1 | B | V_2 \rangle$

- In QM we can estimate exp. values $\langle \psi | A | \psi \rangle$; but what if we want to estimate $\langle \phi | A | \psi \rangle$? $\langle \phi | A | \psi \rangle = \frac{1}{4} \sum_{k=0}^{3} i^k \langle \psi + i^k \phi | A | \psi + i^k \phi \rangle$

We prepare $|\Psi_k\rangle := \frac{1}{\sqrt{2}} \left(|\psi\rangle + i^k |\phi\rangle \right)$ for $k = 0, 1, 2, 3$

and measure exp. values.

c) Use this to show that the class of operators $A \in L(\mathcal{H})$ which preserve the inner product is exactly the set of unitaries. I.e. if $\forall \psi, \phi : \langle A\psi|A\phi\rangle = \langle\psi|\phi\rangle$ then A is unitary and vice versa.

- A unitary $(A^\dagger A = AA^\dagger = \mathbb{1}) \Rightarrow \langle A\psi|A\phi\rangle = \langle\psi|A^\dagger A|\phi\rangle = \langle\psi|\phi\rangle \quad \forall\psi,\phi.$
- $\langle A\psi|A\phi\rangle = \langle\psi|\phi\rangle \ \forall \psi,\phi$

$$\| \qquad$$

$\langle\psi|A^\dagger A|\phi\rangle$ (Note that $\ker(A) = \{\vec{0}\}$)

$$\Rightarrow A^\dagger A = \mathbb{1} \Rightarrow AA^\dagger A = A \Rightarrow (AA^\dagger - \mathbb{1})A = 0 \Rightarrow (AA^\dagger - \mathbb{1})\underbrace{AA^{-1}}_{\mathbb{1}} = 0\cdot A^{-1} = 0$$

$\exists A^{-1}$ since $\ker(A) = \{\vec{0}\}$

d) A useful property of positive operators is the following: If A is a positive operator then there exists a unique positive operator $A^{1/2}$ which satisfies $A^{1/2}A^{1/2} = A$. Moreover this operator satisfies $[A, H] = 0 \implies [A^{1/2}, H] = 0$. Use this to show that the product of two positive operators is positive if and only if they commute.

- $A \geq 0 \Rightarrow \exists$ unique $B \geq 0$ such that $B^2 = A$. $(\sqrt{A} := B)$

PROOF:

- $A \geq 0 \Rightarrow \exists\, U$ unitary such that $A = UDU^\dagger$, $D \geq 0$ diagonal $\Rightarrow B := UD^{1/2}U^\dagger$ satisfies $B^2 = A$, $B \geq 0$
- B is unique.

If $C \geq 0$ such that $C^2 = A$, then $C|v_i\rangle = \lambda_i|v_i\rangle \Rightarrow A|v_i\rangle = \lambda_i^2|v_i\rangle$ (real and positive.)

The eigenspaces of C are the same of A, and eigenvalues are the $\sqrt{\cdot}$.

- $[A, H] = 0 \Rightarrow \boxed{[\sqrt{A}, H]} = 0$

PROOF:

- $A \geq 0$, $A = \sum_i \lambda_i \Pi_{\lambda_i}$ (Projector on eigenspace V_{λ_i} associated to λ_i.), $\sqrt{A} = \sum_i \sqrt{\lambda_i}\, \Pi_{\lambda_i}$

- $[A, H] = 0 \Rightarrow [\Pi_{\lambda_i}, H] = 0 \quad \forall i$

SUBPROOF:

Let $|v_s\rangle \in V_{\lambda_i}$:

- $A|v_s\rangle = \lambda_s |v_s\rangle \Longrightarrow A(H|v_s\rangle) \overset{[A,H]=0}{=} \lambda_s (H|v_s\rangle) \Rightarrow (H|v_s\rangle) \in V_{\lambda_i}$
- $\Pi_{\lambda_i} H = H \Pi_{\lambda_i} \Leftarrow$ They act on the same way on basis vectors.

(Take as basis the orthonormal eigenvectors of A)

- $\sqrt{A} = \sum_i \sqrt{\lambda_i}\, \Pi_{\lambda_i} \Rightarrow [\sqrt{A}, H] = 0$

- $A \geq 0, B \geq 0$
- $AB \geq 0 \Longleftrightarrow AB = BA$

PROOF:

"$\Leftarrow$" $AB = BA \Rightarrow \exists\, U$ unitary such that
- $A = U D_A U^\dagger$ with $D_A \geq 0$ (diagonal)
- $B = U D_B U^\dagger$ with $D_B \geq 0$

- $AB = U D_A D_B U^\dagger$ with $D_A D_B \geq 0$

The eigenvalues are the elements of $D_A D_B$ which is diagonal and positive $\Rightarrow AB \geq 0$

Alternative proof: $\langle\psi| AB |\psi\rangle = \langle\psi| \sqrt{A}\sqrt{A} B |\psi\rangle \overset{[A,B]=0}{=} \langle\psi| \sqrt{A} B \sqrt{A} |\psi\rangle$

$\overset{\sqrt{A}^\dagger = \sqrt{A}}{=} \left\langle \sqrt{A}|\psi\rangle \right| B \left| \sqrt{A}|\psi\rangle \right\rangle \overset{B \geq 0}{\geq} 0 \Rightarrow AB \geq 0$

"$\Rightarrow$": $AB \geq 0 \Rightarrow AB = (AB)^{+} = B^{+}A^{+} = BA$

$(C \geq 0 \Rightarrow C^{+} = C)$ $\quad$ $(C \geq 0 \Rightarrow C^{+} = C)$

e) Even though the product of two positive operators is not necessarily positive, the following holds $A \geq 0 \wedge B \geq 0 \implies \operatorname{Tr} AB \geq 0$. Show this.

$$\operatorname{tr}(AB) = \sum_{i=1}^{d} \underset{\substack{\uparrow \\ \text{eigenvalues of } A}}{a_i} \left(|a_i\rangle \langle \underset{\substack{\uparrow \\ \text{eigenvectors of } A}}{a_i} | B \right) =$$

$$= \sum_{i=1}^{d} \underbrace{a_i}_{\geq 0} \underbrace{\langle a_i | B | a_i \rangle}_{\geq 0} \geq 0$$

Alternative:

$$\operatorname{tr}(AB) = \operatorname{tr}(\sqrt{A}\sqrt{A}B) = \operatorname{tr}(\sqrt{A}B\sqrt{A}) = \operatorname{tr}(\sqrt{A}\sqrt{B}\sqrt{B}\sqrt{A})$$

$$= \operatorname{tr}\Big((\sqrt{B}\sqrt{A})^{+} \underbrace{(\sqrt{B}\sqrt{A})}_{=: C} \Big) = \operatorname{tr}(C^{+}C) \geq 0$$

$C^{+}C \geq 0 \ \forall C.$

2. **Local operations and classical communication (LOCC).**

At the heart of entanglement theory lies the notion of LOCC. To see why, imagine two parties that are a large distance apart from each other, say, Alice is in Berlin and Bob in New York. While they may obtain access to shared entanglement from a third party, it is unreasonable to assume that they are able to perform global operations on the state they share. On the other hand, it is perfectly conceivable that they transmit classical messages, for example, to communicate measurement results.

The goal of this problem is to show that if Alice and Bob are in far away labs, and share a state, any measurement on Alice's part of the state can be simulated as follows: Bob performs a measurement on his side and communicates the result to Alice, who performs a local unitary tranformation. This can be proven for POVMs, but for simplicity we will restrict ourselves to projective measurements.

Consider a bipartite state $|\psi\rangle_{AB}$ with Schmidt decomposition $|\psi\rangle_{AB} = \sum_i \sqrt{\lambda_i}\,|a_i\rangle\,|b_i\rangle$and a projective measurement $\Pi = \{\Pi_i^A\}_i$ acting on Alice's Hilbert space.

a) Expand Π_i^A in the Schmidt basis and define a projective measurement $\Gamma = \{\Gamma_i^B\}_i$ on Bob's system such that the probability p_k^B that Bob observes result k when measuring Γ is the same as the probability p_k^A that Alice observes result k when measuring Π.

- $|\psi\rangle_{AB} := \sum_{i=1}^{\min(d_A,d_B)} \sqrt{\lambda_i}\,|a_i\rangle \otimes |b_i\rangle$

- $\Pi_K^A = \sum_{\substack{i=1\\ j=1}}^{d_A} |a_i\rangle\langle a_i|\Pi_K^A|a_j\rangle\langle a_j|$ (completing the basis eventually).

$\quad = \sum_{i,j} (\Pi_K^A)_{i,j}\,|a_i\rangle\langle a_j|$

- Γ such that $P_K^B = P_K^A$ where $P_K^B = \mathrm{tr}(\rho_B \Gamma_K)$,

$P_K^A = \mathrm{tr}(\rho_A \Pi_K)$,

$\rho_A := \mathrm{tr}_B[|\psi\rangle\langle\psi|]$, $\rho_B := \mathrm{tr}_A[|\psi\rangle\langle\psi|]$

$$P_K^A = \mathrm{tr}(\rho_A \Pi_K^A) = \sum_{l=1}^{\min(d_A,d_B)} \lambda_l \langle a_l|\Pi_K^A|a_l\rangle \underset{\uparrow}{=} \sum_{l=1}^{\min(d_A,d_B)} \lambda_l (\Pi_K^A)_{l,l}$$

$\Pi_K^A = \sum_{i,j} (\Pi_K^A)_{i,j}\,|a_i\rangle\langle a_j|$

$$p_K^B = \mathrm{tr}(\rho_B \Gamma_K^B) = \sum_i \lambda_i \langle b_i | \Gamma_K | b_i \rangle \underset{?}{=} p_K^A$$

If I define $\Gamma_K^B = \sum_{i,j} (\Pi_K^A)_{i,j} \, |b_i\rangle\langle b_j|$

b) Determine the post measurement states $|\phi_j^A\rangle$ after Alice measures Π and obtains result j, and $|\phi_j^B\rangle$ after Bob measures Γ and obtains result j. (both of these states are defined on the whole Hilbert space AB, the superscripts serve to identify who performed the measurement).

$$\rho_{AB}^{(\text{POST ALICE'S OUTCOME "K"})} = \frac{\Pi_K^A \otimes \mathbb{1} \, \rho_{AB} \, \Pi_K^A \otimes \mathbb{1}}{\mathrm{Tr}(\Pi_K^A \otimes \mathbb{1} \, \rho_{AB} \, \Pi_K^A \otimes \mathbb{1})} =$$

$$= \frac{\Pi_K^A \otimes \mathbb{1} \, |\psi\rangle\langle\psi| \, \Pi_K^A \otimes \mathbb{1}}{\mathrm{Tr}(\Pi_K^A \rho_A)} =$$

$$= \frac{\sum_{i,j} (\Pi_K^A)_{i,j} |a_i\rangle\langle a_j| \otimes \mathbb{1} \, \sum_{\ell,x} \sqrt{\lambda_\ell}\sqrt{\lambda_x} |a_\ell\rangle\langle a_x| \otimes |b_\ell\rangle\langle b_x| \, \sum_{i',j'} (\Pi_K^A)_{i',j'} |a_{i'}\rangle\langle a_{j'}| \otimes \mathbb{1}}{\mathrm{Tr}(\Pi_K^A \rho_A)}$$

$$= \frac{\sum_{i,j} (\Pi_K^A)_{i,j} |a_i\rangle\langle a_j| \otimes \mathbb{1} \, \sqrt{\lambda_j}\sqrt{\lambda_{i'}} |a_j\rangle\langle a_{i'}| \otimes |b_j\rangle\langle b_{i'}| \, \sum_{i',j'} (\Pi_K^A)_{i',j'} |a_{i'}\rangle\langle a_{j'}| \otimes \mathbb{1}}{p_K^A}$$

$$= \frac{\sum_{i,j,i',j'} \sqrt{\lambda_j}\sqrt{\lambda_{i'}} (\Pi_K^A)_{i,j} (\Pi_K^A)_{i',j'} \, |a_i\rangle\langle a_{j'}| \otimes |b_j\rangle\langle b_{i'}|}{p_K^A}$$

$$\Rightarrow |\psi\rangle_{AB}^{(\text{POST ALICE'S "K"})} = \frac{\sum_{i,j} (\Pi_K^A)_{i,j} \sqrt{\lambda_j} \, |a_i\rangle \otimes |b_j\rangle}{\sqrt{p_K^A}}$$

$$|\psi\rangle_{AB}^{(\text{POST BOB'S OUTCOME "K"})} = \frac{\sum_{i,j} (\Pi_K^A)_{i,j} \sqrt{\lambda_j} \, |a_j\rangle \otimes |b_i\rangle}{\sqrt{p_K^A}} = \frac{\sum_{i,j} (\Pi_K^A)_{j,i} \sqrt{\lambda_i} \, |a_i\rangle \otimes |b_j\rangle}{\sqrt{p_K^A}}$$

c) Show that $|\phi_j^A\rangle$ and $|\phi_j^B\rangle$ are equivalent up to local unitary transformations.

$$|\psi\rangle_{AB}^{(\text{POST ALICE'S "k"})} = \sum_{i,s} (C)_{i,s} |a_i\rangle \otimes |b_s\rangle$$

$$= \sum_l D_{ll} \underbrace{\left(\sum_i U_{il}|a_i\rangle\right)}_{|\tilde{a}_l\rangle = U|a_l\rangle \text{ ← orthonorm.}} \otimes \underbrace{\left(\sum_s (V^\dagger)_{ls} |b_s\rangle\right)}_{|\tilde{b}_l\rangle = V^*|b_l\rangle \text{ ← orthonormal.}}$$

$C \underset{\text{SVD}}{=} UDV^\dagger \Rightarrow (C)_{i,s} = \sum_l U_{il} D_{ll} V^\dagger_{ls}$

$$= \sum_l D_{ll} |\tilde{a}_l\rangle \otimes |\tilde{b}_l\rangle$$

$$|\psi\rangle_{AB}^{(\text{POST BOB'S "k"})} = \sum_{i,s} (C)_{s,i} |a_i\rangle \otimes |b_s\rangle$$

$$= \sum_l D_{ll} |\tilde{a}'_l\rangle \otimes |\tilde{b}'_l\rangle =$$

- $(C)_{i,s} = \sum_l U_{il} D_{ll} V^\dagger_{ls}$
- Define $|\tilde{a}'_l\rangle$ and $|\tilde{b}'_l\rangle$ as before similarly.

$$= U_A \otimes U_B \sum_l D_{ll} |\tilde{a}_l\rangle \otimes |\tilde{b}_l\rangle =$$

U_A s.t. $|\tilde{a}'_l\rangle = U_A|\tilde{a}_l\rangle$

U_B s.t. $|\tilde{b}'_l\rangle = U_B|\tilde{b}_l\rangle$

$$= U_A \otimes U_B |\psi\rangle_{AB}^{(\text{POST ALICE "k"})}$$

$$|\psi\rangle_{AB}^{(\text{POST ALICE "k"})} = U_A^\dagger \otimes U_B^\dagger |\psi\rangle_{AB}^{(\text{POST BOB'S "k"})}$$

d) Describe the LOCC protocol.

- BOB performs measurement Γ and she gets "K", then he calls ALICE and communicates "K".
- They both apply local unitaries to change the state from

$|\psi\rangle_{AB}^{(\text{POST BOB'S "K"})}$ in $|\psi\rangle_{AB}^{(\text{POST ALICE'S "K"})}$.

Freie Universität Berlin

Tutorials on Quantum Information Theory

Winter term 2022/23

Problem Sheet 6

Operator Properties and LOCC

J. Eisert, A. Townsend-Teague, A. Mele, A. Burchards, J. Denzler

1. **Operator Properties**

In this exercise we take a short break from following the main content covered in the lecture and return back to proving some simple but useful identities for operators on complex Hilbert spaces. In particular, we explore the two important facts that operators are completely specified by their diagonal elements in all bases as well as the power of the square root representation for positive operators.

a) Interestingly in a complex inner product space an operator is fully specified when its diagonal elements in all bases are known. Show this by verifying the identity

$$\langle\psi|\, A\,|\phi\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \left\langle \psi + i^k\phi\right|\, A\,\left|\psi + i^k\phi\right\rangle . \tag{1}$$

b) Use the previous identity to show that $\forall\psi : \langle\psi|\, A\,|\psi\rangle = \langle\psi|\, B\,|\psi\rangle \implies A = B$.

c) Use this to show that the class of operators $A \in L(\mathcal{H})$ which preserve the inner product is exactly the set of unitaries. I.e. if $\forall\psi,\phi : \langle A\psi|A\phi\rangle = \langle\psi|\phi\rangle$ then A is unitary and vice versa.

d) A useful property of positive operators is the following: If A is a positive operator then there exists a unique positive operator $A^{1/2}$ which satisfies $A^{1/2}A^{1/2} = A$. Moreover this operator satisfies $[A, H] = 0 \implies [A^{1/2}, H] = 0$. Use this to show that the product of two positive operators is positive if and only if they commute. (hint: Also show that $A \geq B \wedge B \geq A \implies A = B$).

e) Even though the product of two positive operators is not necessarily positive, the following holds $A \geq 0 \wedge B \geq 0 \implies \operatorname{Tr} AB \geq 0$. Show this.

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c) Show that $\left|\phi_j^A\right\rangle$ and $\left|\phi_j^B\right\rangle$ are equivalent up to local unitary transformations.

d) Describe the LOCC protocol.