1. **Examples of quantum channels** (9 Points: 1+4+2+2)
Now we are ready to look at some examples of quantum channels acting on qubits, i.e., $\mathcal{H} = \mathbb{C}^2$. The following maps are important so-called noise channels

$$F_\epsilon(A) := \epsilon XAX + (1-\epsilon)A$$
$$D_\epsilon(A) := \epsilon \operatorname{Tr}[A]\frac{\mathbb{1}}{d} + (1-\epsilon)A$$
$$A_\epsilon(A) := \epsilon \operatorname{Tr}[A]\,|0\rangle\langle 0| + (1-\epsilon)A,$$

where $\epsilon \in [0,1]$.

a) For each channel, show that it is CPT.

Note first of all that they are all trace-preserving $tr(\phi(\rho)) = tr(\rho)$ and linear: $\phi(\alpha\rho_1 + \beta\rho_2) = \alpha\phi(\rho_1) + \beta\phi(\rho_2)$

• CPT MAPS = $\underbrace{\text{COMPLETELY POSITIVE}}_{\star_1}$ $\underbrace{\text{TRACE-PRESERVING}}_{\star_2}$ $\underbrace{\text{(LINEAR)}}_{\star_3}$ MAPS

• LINEARITY: $\star_3$ We need to show that:

$$\phi(\alpha\rho_1 + \beta\rho_2) = \alpha\,\phi(\rho_1) + \beta\,\phi(\rho_2)$$

for $\phi(\cdot) = \underbrace{F_\varepsilon(\cdot),\ D_\varepsilon(\cdot),\ A_\varepsilon(\cdot)}$.

OK, THIS IS EASY! ☺

• TRACE PRESERVING: $\star_2$ We need to show that:

$$tr(\phi(\rho)) = tr(\rho) \quad \forall \rho \in S(\mathcal{H})$$

• $Tr(F_\varepsilon(A)) \underset{\uparrow}{=} \varepsilon\, Tr(XXA) + (1-\varepsilon)\,Tr(A) \underset{\substack{\uparrow \\ X^2 = \mathbb{1}}}{=} Tr(A)$

$F_\epsilon(A) := \epsilon XAX + (1-\epsilon)A$

• $Tr(D_\varepsilon(A)) \underset{?}{=} \varepsilon\, Tr[A]\underbrace{\frac{Tr[\mathbb{1}]}{d}}_{=1} + (1-\varepsilon)\,Tr[A] = Tr[A]$

$D_\epsilon(A) := \epsilon \operatorname{Tr}[A]\frac{\mathbb{1}}{d} + (1-\epsilon)A$

- $\mathrm{Tr}[A_\varepsilon(A)] = \varepsilon \mathrm{Tr}[A] \underbrace{\mathrm{Tr}[|b\rangle\langle a|]}_{=1} + (1-\varepsilon)\mathrm{Tr}[A] = \mathrm{Tr}[A]$

$$A_\epsilon(A) := \epsilon \, \mathrm{Tr}[A] \, |0\rangle\langle 0| + (1-\epsilon)A,$$

- COMPLETELY POSITIVE: (i.e. Φ compl. positive $\overset{\text{DEF}}{\iff}$ $\Phi \otimes \mathbb{1}(\sigma) \geq 0 \quad \forall \sigma \in S(\mathcal{H} \otimes \mathcal{H}_B)$)

To show it, we'll use one of the most important Q-CHANNEL theorems:

CHOI STATE THEOREM

$\underset{\text{CHOI STATE}}{P_\Phi} := \Phi \otimes \mathbb{1}_{\mathcal{H}} [|\Omega\rangle\langle\Omega|] \geq 0$ (where $|\Omega\rangle := \sum_{i=1}^{d} \frac{|i\rangle \otimes |i\rangle}{\sqrt{d}}$) $\Longrightarrow$ Φ completely positive.

- We'll use also a little LEMMA:

LEMMA: Φ_1, Φ_2 Q-CHANNEL $\Rightarrow$ $\varepsilon \Phi_1(\cdot) + (1-\varepsilon)\Phi_2(\cdot)$ with $\varepsilon \in [0,1]$ is Q-CHANNEL.
PROOF: EASY! $\boxed{\smile}$

This LEMMA can be generalized: a CONVEX COMBINATION OF Q-CHANNELS IS A Q-CHANNEL.

$$F_\epsilon(A) := \underbrace{\epsilon XAX}_{1} + \underbrace{(1-\epsilon)A}_{2}$$

because of the LEMMA, we treat them separately.

1) $\cdot \; X(\cdot)X \otimes \mathbb{1} \underbrace{[|\Omega\rangle\langle\Omega|]}_{=\frac{1}{d}\sum_{i,j}|i\rangle\langle j|\otimes|i\rangle\langle j|} = \frac{1}{d}\sum_{i,j=1}^{d} X|i\rangle\langle j|X \otimes |i\rangle\langle j| \overset{?}{\geq} 0$

$$\langle\psi| \frac{1}{d}\sum_{i,j=1}^{d} X|i\rangle\langle j|X \otimes |i\rangle\langle j| \, |\psi\rangle \underset{X^\dagger = X}{=} \left(\langle\psi| X^\dagger \otimes \mathbb{1}\right) |\Omega\rangle\langle\Omega| \left(X \otimes \mathbb{1}|\psi\rangle\right) \underset{|\psi'\rangle = X\otimes\mathbb{1}|\psi\rangle}{=} \langle\psi'|\Omega\rangle\langle\Omega|\psi'\rangle$$

$$= |\langle\psi'|\Omega\rangle|^2 \geq 0$$

2) $\cdot \; (\cdot)\,|\Omega\rangle\langle\Omega| = |\Omega\rangle\langle\Omega| \geq 0$

$$D_\epsilon(A) := \epsilon \,\mathrm{Tr}[A]\frac{\mathbb{1}}{d} + (1-\epsilon)A$$

CP.

$$\bullet\quad \mathrm{Tr}[\cdot]\frac{\mathbb{1}}{d} \otimes \mathbb{1}\left(|\Omega\rangle\langle\Omega|\right) = \frac{1}{d}\sum_{i,j} \mathrm{Tr}\left[|i\rangle\langle j|\right]\frac{\mathbb{1}}{d} \otimes |i\rangle\langle j| =$$

$$= \frac{1}{d}\sum_{i,j} \delta_{i,j}\frac{\mathbb{1}}{d} \otimes |i\rangle\langle j| =$$

$$= \frac{\mathbb{1}}{d} \otimes \frac{1}{d}\sum_{i} |i\rangle\langle i| = \frac{\mathbb{1}}{d} \otimes \frac{\mathbb{1}}{d} \geq 0$$

$$A_\epsilon(A) := \epsilon \,\mathrm{Tr}[A]\,|0\rangle\langle 0| + (1-\epsilon)A,$$

CP

$$\mathrm{Tr}[\cdot]\,|0\rangle\langle 0| \otimes \mathbb{1}\left(|\Omega\rangle\langle\Omega|\right) = \frac{1}{d}\sum_{i,j} \delta_{i,j}\,|0\rangle\langle 0| \otimes |i\rangle\langle j| =$$

AS BEFORE

$$= |0\rangle\langle 0| \otimes \frac{\mathbb{1}}{d} \geq 0$$

$\rho_1 \otimes \rho_2$ state if ρ_1, ρ_2 state.

SUBPROOF:

- $\mathrm{Tr}(\rho_1 \otimes \rho_2) = \underbrace{\mathrm{Tr}[\rho_1]}_{=1}\underbrace{\mathrm{Tr}[\rho_2]}_{=1} = 1$
- $\rho_1 \otimes \rho_2 = \sum_{i,j} \underbrace{\lambda_i^{(1)}}_{\geq 0}\underbrace{\lambda_j^{(2)}}_{\geq 0} |\lambda_i^{(1)}\rangle\langle\lambda_i^{(1)}| \otimes |\lambda_j^{(2)}\rangle\langle\lambda_j^{(2)}|$

ρ_1 eigen.

ρ_2 eigen.

$\Rightarrow$ eigenvalue $\geq 0 \Rightarrow \rho_1 \otimes \rho_2 \geq 0$

Next, we represent each quantum channel in different three ways, discussed in the previous exercise.

b) For each channel with fixed $\epsilon = 1$, give its Choi-Jamiołkowski state, a Kraus representation and a Stinespring representation.

Remember from my last tutorial that:

CHOI STATE: $\rho_\Phi := \Phi \otimes \mathbb{1}\left[|\Omega\rangle\langle\Omega|\right]$

KRAUS: $\boxed{\Phi(\rho) = \sum_{i=1}^{d^2} K_i \rho K_i^\dagger \quad \text{with} \quad \sum_{i=1}^{d^2} K_i^\dagger K_i = \mathbb{1}}$; for example $K_i|\cdot\rangle = {}_B\langle \cdot^* | \underset{\substack{\uparrow \\ \rho_\Phi \text{ eigenstate}}}{\Psi_i\rangle_{AB}} \underset{\text{eigenvalue of } \rho_\Phi}{\sqrt{p_i}} \sqrt{d}$

$\left(\text{i.e.: } \rho_\Phi = \sum_{i=1}^{d^2} p_i |\Psi_i\rangle_{AB}\langle\Psi_i|_{AB} \right)$

$\Rightarrow K_i = \sum_{j=1}^{d} K_i \underset{\substack{\uparrow \\ \text{comp. basis}}}{|j\rangle}\langle j| = \sqrt{p_i}\sqrt{d} \sum_{j=1} {}_B\langle j|\Psi_i\rangle_{AB}\langle j|$

STINESPRING: $\Phi(\rho) = \mathrm{tr}_B\left(U_{AB}\, \rho \otimes |0\rangle\langle 0|\, U_{AB}^\dagger\right)$, e.g. $U_{AB}|\cdot\rangle\otimes|0\rangle = \sum_{i=1}^{d^2} K_i|\cdot\rangle\otimes|i\rangle$

$$F_\epsilon(A) := \epsilon XAX + (1-\epsilon)A \quad = \quad XAX$$
$$D_\epsilon(A) := \epsilon\,\mathrm{Tr}[A]\frac{\mathbb{1}}{d} + (1-\epsilon)A \quad = \quad \mathrm{Tr}[A]\frac{\mathbb{1}}{d}$$
$$A_\epsilon(A) := \epsilon\,\mathrm{Tr}[A]\,|0\rangle\langle 0| + (1-\epsilon)A, \quad \underset{\substack{\uparrow \\ \epsilon=1}}{=} \quad \mathrm{Tr}[A]\,|0\rangle\langle 0|$$

- CHOI STATE:

- $F_{\epsilon=1}(\cdot)\otimes\mathbb{1}\left[|\Omega\rangle\langle\Omega|\right] = X(\cdot)X\otimes\mathbb{1}\left[|\Omega\rangle\langle\Omega|\right]$

$$\underset{\substack{\uparrow \\ \text{BEFORE}}}{=} \frac{1}{d}\sum_{i,j=1}^{d} X|i\rangle\langle j|X\otimes|i\rangle\langle j| \underset{\substack{\uparrow \\ d=2}}{=} \frac{1}{2}\sum_{i,j\in\{0,1\}} |\bar{i}\rangle\langle\bar{j}|\otimes|i\rangle\langle j|$$

$$= \left(\frac{|01\rangle + |10\rangle}{\sqrt{2}}\right)\left(\frac{\langle 01| + \langle 10|}{\sqrt{2}}\right) =: |\Omega_1\rangle\langle\Omega_1|$$

- $D_{\epsilon=1}(\cdot)\otimes\mathbb{1}\left[|\Omega\rangle\langle\Omega|\right] = \mathrm{Tr}[\cdot]\frac{\mathbb{1}}{d}\otimes\mathbb{1}\left[|\Omega\rangle\langle\Omega|\right]$

$$= \frac{\mathbb{1}}{d}\otimes\frac{\mathbb{1}}{d}$$

- $A_{\varepsilon=1}(\cdot) \otimes \mathbb{1}\,[\,|\Omega\rangle\langle\Omega|\,] = \mathrm{Tr}[\cdot]\,|0\rangle\langle 0| \otimes \mathbb{1}\,[\,|\Omega\rangle\langle\Omega|\,]$

$$= |0\rangle\langle 0| \otimes \frac{\mathbb{1}}{d}$$

- ~~KRAUS~~ $\quad \Phi(\rho) = \sum_{i=1}^{d^2} K_i\, \rho\, K_i^{+}$ where $\quad K_i = \sum_{j=1}^{d} K_i |j\rangle\langle j| = \sqrt{p_i}\sqrt{d} \sum_{j=1} {}_B\langle j|\Psi_i\rangle_{AB} \langle j| \equiv \textcircled{\star}$

  comp. basis

  with $\sum_i K_i^{+} K_i = \mathbb{1}$

  ρ_Φ eigenstate, eigenvalue of ρ_Φ

  (i.e.: $\rho_\Phi = \sum_{i=1}^{d^2} p_i |\Psi_i\rangle_{AB}\langle\Psi_i|_{AB}$)

- $F_{\varepsilon=1}(\rho) = X \rho X \;\rightsquigarrow\;$ KRAUS $\quad K_1 = X \,,\; K_{i\geq 1} = 0$

  - $\sum_i K_i^{+} K_i = X^{+} X = X X = \mathbb{1}$

- $D_{\varepsilon=1}(\rho) = \mathrm{Tr}[\rho]\,\dfrac{\mathbb{1}}{d}$

$$\rho_{D_{\varepsilon=1}} = \underbrace{\frac{1}{d^2}\,\mathbb{1}\otimes\mathbb{1}}_{\{|i\rangle\otimes|j\rangle,\; i,j=0,1\}\text{ eigenstates}} \;\rightsquigarrow\; K_{(i,j)} \underset{\textcircled{\star}}{=} \frac{\sqrt{2}}{\sqrt{4}} \sum_{l=1} {}_B\langle l|i,j\rangle_{AB}\langle l| = \frac{1}{\sqrt{2}}|i\rangle\langle j|$$

- $\sum_i K_i^{+} K_i = \frac{1}{2}\sum_{i,j} (|i\rangle\langle j|)^{+}(|i\rangle\langle j|) = \mathbb{1}$

$$\boxed{\begin{aligned} \text{CHECK}:\; D_1(\rho) &= \sum_{(i,j)\in\{0,1\}} K_{(i,j)}\,\rho\,K_{(i,j)}^{+} = \\ &= \sum_{(i,j)\in\{0,1\}} \frac{1}{2}|i\rangle\langle j|\,\rho\,|j\rangle\langle i| = \frac{\mathrm{Tr}[\rho]}{2} = \frac{\mathrm{Tr}[\rho]}{d} \end{aligned}}$$

Other equiv. KRAUS OPERATORS are: $\frac{1}{2}\{\mathbb{1}, X, Y, Z\}$.

- $A_{\varepsilon=1}(\rho) = \mathrm{Tr}[A]\,|0\rangle\langle 0|$

$$\rho_{A_{\varepsilon=1}} = \underbrace{|0\rangle\langle 0| \otimes \frac{\mathbb{1}}{d}}_{\text{(eigenstates: } |0\rangle\otimes|i\rangle \text{ with } i=0,1)} \;\rightsquigarrow\; K_i \underset{\textcircled{\star}}{=} \sum_{l=1}^{d} \sqrt{p_i}\sqrt{d}\; {}_B\langle l|\Psi_i\rangle_{AB}\langle l| = \sum_{l=1}^{d} \sqrt{\frac{1}{d}}\sqrt{d}\; {}_B\langle l|0,i\rangle_{AB}\langle l| =$$

$$= |0\rangle\langle i|$$

CHECK:

$$A_{\varepsilon=1}(\rho) = \sum_{i=1}^{d^2} K_i \rho K_i^{\dagger} = \sum_{i=1}^{d} |0\rangle\langle i|\rho|i\rangle\langle 0| = \mathrm{Tr}[\rho]\,|0\rangle\langle 0|$$

• STINESPRING: $\Phi(\rho) = tr_B\left(U_{AB}\ \rho\otimes|0\rangle\langle 0|\ U_{AB}^{\dagger}\right)$ where $U_{AB}|\cdot\rangle\otimes|0\rangle = \sum_{i=1}^{d^2} K_i|\cdot\rangle\otimes|i\rangle$ $\Leftarrow$ (★A)

• $F_{\varepsilon=1}(\rho) = X\rho X \rightsquigarrow F_{\varepsilon=1}(\rho) = tr_B\left(X\otimes\mathbb{1}\ \rho\otimes|0\rangle\langle 0|\ X^{\dagger}\otimes\mathbb{1}\right)$

$\uparrow$ $X = X^{\dagger}$

$$\exp(A) = \sum_{i=0}^{\infty}\frac{A^i}{i!}$$

$$U_{AB} = X\otimes\mathbb{1} = (-i)\,e^{i(X\otimes\mathbb{1})\frac{\pi}{2}}$$

LEMMA $\boxed{☺}$: P such that $P^2=\mathbb{1} \Rightarrow e^{iPt} = \cos(t)\,\mathbb{1} + i\sin(t)\,P$

PROOF:

$$e^{iPt} := \sum_{k=0}^{\infty}\frac{(iPt)^k}{k!} \underset{\text{even/odd splitting}}{=} \sum_{k=0}^{\infty}\frac{(iPt)^{2k}}{(2k)!} + \sum_{k=0}^{+\infty}\frac{(iPt)^{2k+1}}{(2k+1)!} =$$

$$= \sum_{k=0}^{+\infty}\frac{i^{2k}P^{2k}t^{2k}}{(2k)!} + \sum_{k=0}^{+\infty}\frac{i^{2k+1}P^{2k+1}t^{2k+1}}{(2k+1)!} =$$

$$\underset{\substack{\cdot P^2=\mathbb{1}\\ \cdot i^2=-1}}{=} \underbrace{\left(\sum_{k=0}^{+\infty}\frac{(-1)^k t^{2k}}{(2k)!}\right)}_{\cos(t)}\mathbb{1} + iP\underbrace{\left(\frac{(-1)^k t^{2k+1}}{(2k+1)!}\right)}_{\sin(t)}$$

• $D_{\varepsilon=1}(\rho) = \mathrm{Tr}[\rho]\,\frac{\mathbb{1}}{d} = \sum_{(i,j)\in\{0,1\}} \underbrace{\left(\frac{|i\rangle\langle j|}{\sqrt{2}}\right)}_{K_{(i,j)}}\rho\left(\frac{|j\rangle\langle i|}{\sqrt{2}}\right)$

(★A) $\rightarrow \Phi(\rho) = tr_B\left(U_{AB}\ \rho\otimes|0\rangle\langle 0|\ U_{AB}^{\dagger}\right)$, e.g. $U_{AB}|\cdot\rangle\otimes|0\rangle = \sum_{i=1}^{d^2} K_i|\cdot\rangle\otimes|i\rangle$

$$U_{AB}|\ell\rangle\otimes|0\rangle = \sum_{i=1}^{d^2} K_i|\ell\rangle\otimes|i\rangle =$$

(RELABELLING) $\rightarrow$ $= K_1|\ell\rangle\otimes|1\rangle + K_2|\ell\rangle\otimes|2\rangle + K_3|\ell\rangle\otimes|3\rangle + K_4|\ell\rangle\otimes|4\rangle$

$$= \frac{|0\rangle\langle 0|\ell\rangle}{\sqrt{2}}\otimes|1\rangle + \frac{|0\rangle\langle 1|\ell\rangle}{\sqrt{2}}\otimes|2\rangle + \frac{|1\rangle\langle 0|\ell\rangle}{\sqrt{2}}\otimes|3\rangle + \frac{|1\rangle\langle 1|\ell\rangle}{\sqrt{2}}\otimes|4\rangle$$

$$U_{AB}|0\rangle\otimes|0\rangle = \frac{|0\rangle\otimes|1\rangle + |1\rangle\otimes|3\rangle}{\sqrt{2}} =: |\Omega_0\rangle$$

$$U_{AB}|1\rangle\otimes|0\rangle = \frac{|0\rangle\otimes|2\rangle + |1\rangle\otimes|4\rangle}{\sqrt{2}} =: |\Omega_1\rangle$$

$$\Rightarrow\ U_{AB}|\psi\rangle\otimes|0\rangle \underset{\uparrow}{=} \langle 0|\psi\rangle|\Omega_0\rangle + \langle 1|\psi\rangle|\Omega_1\rangle$$

$$|\psi\rangle = |0\rangle\langle 0|\psi\rangle + |1\rangle\langle 1|\psi\rangle$$

$$\left(U_{AB}|\psi\rangle\otimes|0\rangle\right)\left(\langle\psi|\otimes\langle 0|U_{AB}^{\dagger}\right) = \left(\langle 0|\psi\rangle|\Omega_0\rangle + \langle 1|\psi\rangle|\Omega_1\rangle\right)\left(\langle\psi|0\rangle\langle\Omega_0| + \langle\psi|1\rangle\langle\Omega_1|\right)$$

$$U_{AB}\,\rho\otimes|0\rangle\langle 0|\,U_{AB}^{\dagger} \underset{\uparrow}{=} \langle 0|\rho|0\rangle|\Omega_0\rangle\langle\Omega_0| + \langle 0|\rho|1\rangle|\Omega_0\rangle\langle\Omega_1| + \langle 1|\rho|0\rangle|\Omega_1\rangle\langle\Omega_0| + \langle 1|\rho|1\rangle|\Omega_1\rangle\langle\Omega_1|$$

$$\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$$

CHECK: $tr_B\left(U_{AB}\,\rho\otimes|0\rangle\langle 0|\,U_{AB}^{\dagger}\right) = \langle 0|\rho|0\rangle\left(\frac{|0\rangle\langle 0|}{2} + \frac{|1\rangle\langle 1|}{2}\right) + 0 + 0 + \langle 1|\rho|1\rangle\left(\frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2}\right) = \frac{tr[\rho]}{2}\mathbb{1}$

- $A_{\varepsilon=1}(\rho) = Tr[\rho]\,|0\rangle\langle 0| = \sum_{i=1}^{d} \underbrace{\left(|0\rangle\langle i|\right)}_{K_i}\rho\left(|i\rangle\langle 0|\right)$

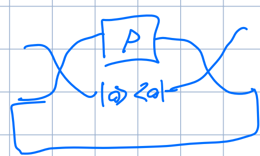
(☆) $\rightarrow \Phi(\rho) = tr_B\left(U_{AB}\,\rho\otimes|0\rangle\langle 0|\,U_{AB}^{\dagger}\right)$ where $U_{AB}|\cdot\rangle\otimes|0\rangle = \sum_{i=1}^{d^2} K_i|\cdot\rangle\otimes|i\rangle$

$$U_{AB}|j\rangle\otimes|0\rangle = \sum_{i=1}^{d}|0\rangle\langle i|j\rangle\otimes|i\rangle = |0\rangle\otimes|j\rangle$$

$$\Rightarrow\ U_{AB}\left(|\psi\rangle\otimes|0\rangle\right) = \sum_{j=1}^{d}\langle j|\psi\rangle\underbrace{U_{AB}|j\rangle\otimes|0\rangle}_{|0\rangle\otimes|j\rangle} = |0\rangle\otimes|\psi\rangle$$

- I can take $U_{AB} = \mathbb{F}$ $\quad\left(\mathbb{F}|i\rangle\otimes|j\rangle := |j\rangle\otimes|i\rangle\right)$

- CHECK: $tr_B\left(\mathbb{F}\,\rho\otimes|0\rangle\langle 0|\,\mathbb{F}\right) =$



$=$

$= Tr[\rho]\,|0\rangle\langle 0|$

$= A_{\varepsilon=1}(\rho)$

$tr_2(A)$ $\quad \rho\otimes|0\rangle\langle 0|$ $\quad \mathbb{F}|\psi\rangle\otimes|\phi\rangle = |\phi\rangle\otimes|\psi\rangle$

- CURIOSITY: • $\mathbb{F} = \frac{1}{d}\sum_{P_i\in\{\mathbb{1},X,Y,Z\}} P_i\otimes P_i$
  - HILBERT-SCHMIDT SCALAR PRODUCT $\Rightarrow \mathbb{F} = \sum_{i,j} c_{i,j}\,P_i\otimes P_j \Rightarrow tr\left(P_i\otimes P_j\,\mathbb{F}\right) \underset{\uparrow\ \text{H.S. BASIS}}{=} c_{i,j}\,d^2$, $\quad tr(P_iP_j) = \delta_{i,j}d$
  - $P_i\otimes P_j$ is a basis

- $U_{AB} = \mathbb{F} \underset{\uparrow}{=} \left(e^{i\mathbb{F}\frac{\pi}{2}}\right)(-i) \underset{\uparrow\ d=2}{=} (-i)\,e^{i\frac{\pi}{4}(\mathbb{1} + X\otimes X + Y\otimes Y + Z\otimes Z)}$

  XXX- ISOTROPIC-HEISENBERG EVOLUTION!

  - $\mathbb{F}^2 = \mathbb{1}$
  - LEMMA ☺

c) Generalise the previous results to an arbitrary $\epsilon \in [0,1]$. (Hint: First compute the respective representations for $\epsilon = 0$ and then reason for the Choi state on the one hand, and for the Kraus and Stinespring representations on the other hand how to combine the $\epsilon = 0$ and $= 1$ cases into an arbitrary ϵ case.)

- First for $\varepsilon = 0$:
  - All of them are equal to $\Phi(\rho) = \rho$
  - So their Choi-state is $\Phi(\cdot) \otimes \mathbb{1}(|\Omega\rangle\langle\Omega|) = |\Omega\rangle\langle\Omega|$
  - A KRAUS decomposition is: $K_1 = \mathbb{1} \rightsquigarrow \Phi_\rho = \sum_i K_i \rho K_i^\dagger = \mathbb{1} \rho \mathbb{1}^\dagger$
  - A STINESPING repr. is: $\Phi(\rho) = tr_B\left(U_{AB}\, \rho \otimes |0\rangle\langle 0|\, U_{AB}^\dagger\right)$

    with $U_{AB} = \mathbb{1}$.

- General $\varepsilon \in [0,1]$

  We have $\Phi_\varepsilon(\rho) = \varepsilon\, \Phi_{\varepsilon=1}(\rho) + (1-\varepsilon)\, \Phi_{\varepsilon=0}(\rho)$

  - The Choi state is $\Phi_\varepsilon \otimes \mathbb{1}(\underset{|\Omega\rangle\langle\Omega|}{\underset{\|}{\Omega}}) = \varepsilon\, \Phi_{\varepsilon=1} \otimes \mathbb{1}(\Omega) + (1-\varepsilon) \underbrace{\Phi_{\varepsilon=0} \otimes \mathbb{1}(\Omega)}_{= |\Omega\rangle\langle\Omega|}$
  - The KRAUS decomposition is $\Phi_\varepsilon(\rho) = \sum_i \tilde{K}_i \rho \tilde{K}_i^\dagger$

    where $\{\tilde{K}_i\} = \left\{ \underbrace{\{\sqrt{1-\varepsilon}\, K_i|_{\varepsilon=0}\}_{i=1,..}}_{= \{\sqrt{1-\varepsilon}\, \mathbb{1}\}},\ \{\sqrt{\varepsilon}\, K_i|_{\varepsilon=1}\}_{i=1,\dots} \right\}$
  - The STINES. repr. is:

    $\Phi(\rho) = tr_B\left(U_{AB}\, \rho \otimes |0\rangle\langle 0|\, U_{AB}^\dagger\right)$ where $U_{AB}|\cdot\rangle \otimes |0\rangle = \sum_i \tilde{K}_i |\cdot\rangle \otimes |i\rangle$

d) For arbitrary $\epsilon \in [0,1]$ compute the action of each channel on the inputs $|0\rangle\langle 0|$ and $\rho = \mathbb{1}/2$. What is the physical interpretation of each channel?

$$F_\epsilon(A) := \epsilon XAX + (1-\epsilon)A$$
$$D_\epsilon(A) := \epsilon \operatorname{Tr}[A]\frac{\mathbb{1}}{d} + (1-\epsilon)A$$
$$A_\epsilon(A) := \epsilon \operatorname{Tr}[A]\,|0\rangle\langle 0| + (1-\epsilon)A,$$

- $F_\varepsilon(|0\rangle\langle 0|) = \varepsilon |1\rangle\langle 1| + (1-\varepsilon)|0\rangle\langle 0|$

BIT - FLIP ERROR

$F_\varepsilon\left(\frac{\mathbb{1}}{2}\right) = \varepsilon \frac{\mathbb{1}}{2} + (1-\varepsilon)\frac{\mathbb{1}}{2} = \frac{\mathbb{1}}{2}$

- $D_\varepsilon(|0\rangle\langle 0|) = \varepsilon \frac{\mathbb{1}}{2} + (1-\varepsilon)|0\rangle\langle 0|$

depolarizing - channel.
($\varepsilon \to 1 \Rightarrow \to$ maximally mixed state)

$D_\varepsilon\left(\frac{\mathbb{1}}{2}\right) = \frac{\mathbb{1}}{2}$

- $A_\varepsilon(|0\rangle\langle 0|) = \varepsilon |0\rangle\langle 0| + (1-\varepsilon)|0\rangle\langle 0| = |0\rangle\langle 0|$

$A_\varepsilon\left(\frac{\mathbb{1}}{2}\right) = \varepsilon |0\rangle\langle 0| + (1-\varepsilon)\frac{\mathbb{1}}{2}$

Reset in the 0- state.

INTERLUDE :

$$\boxed{\text{CLASSICAL INFORMATION THEORY}}$$

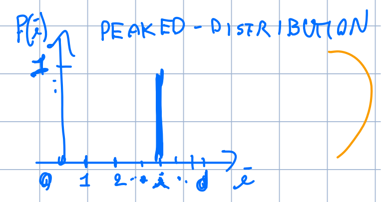
- $X = \{x_1, \ldots, x_d\}$ Random Variable $\overset{\text{DEF.}}{\iff} \begin{cases} \cdot\ P_i \geq 0 \\ \cdot\ \sum_{i=1}^{d} P_i = 1 \end{cases}$

$x_1 \downarrow P_1$, $x_d \downarrow P_d$

Probability to extract x_1

- **DEF:** (SHANNON ENTROPY of X) $\quad H(X) := -\sum_{i=1}^{d} P_i \log(P_i)$

- $H(X) \geq 0 \quad \left(H(X) = 0 \iff \exists\, J \in \{1, \ldots, d\} : P_i = \delta_{i,J} \right)$



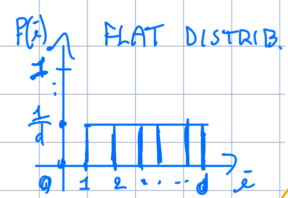
PROOF:

- $0 \leq P_i \leq 1 \Rightarrow -P_i \log(P_i) \geq 0 \Rightarrow H(X) \geq 0$
- $H(X) = 0 \Rightarrow P_i \log(P_i) = 0 \ \forall i$, with $P_i \geq 0$ and $\sum_{i=1}^{d} P_i = 1$

$\Rightarrow P_i = 0$ or $\log(P_i) = 0 \Rightarrow \exists\, J : P_i = \delta_{i,J} \ \forall i.$

$\log(P_i) = 0 \Updownarrow P_i = 1$

- $H(X) \leq \log(d) \quad \left(H(X) = \log(d) \iff P_i = \frac{1}{d} \ \forall i \right)$



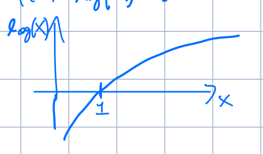
PROOF:

- DEF: $f(x)$ is CONVEX (CONCAVE) $\iff f\left(\sum_{i=1}^{d} P_i x_i\right) \leq (\geq) \sum_{i=1}^{d} P_i f(x_i)$ where $\cdot\ x_i \in \mathbb{R}$, $\cdot\ P_i \in [0,1]$, $\cdot\ \sum_{i=1}^{d} P_i = 1$

- $H(X) = \sum_{i=1}^{d} P_i \left(-\log(P_i)\right) = \sum_{i=1}^{d} P_i \left(\log\left(\frac{1}{P_i}\right)\right) =$

$\leq \log\left(\sum_{i=1}^{d} P_i \cdot \frac{1}{P_i}\right) = \log(d)$

- $f(x) = \log(x)$ is CONCAVE



- "$\Rightarrow$": If $p_i = \frac{1}{d}$ $\forall i$ $\Rightarrow$ $H(X) = \log(d)$

- "$\Leftarrow$": The inequality is saturated if $f\left(\sum_i p_i x_i\right) = \sum_i p_i f(x_i)$ with $x_i = \frac{1}{p_i}$.
  - Now $f(x) = -\log(x)$ is strictly CONCAVE: $\sum_i p_i f(x_i) = f\left(\sum_i p_i x_i\right) \Leftrightarrow x_i = \bar{x}$ $\forall i$.

  $\Rightarrow \frac{1}{p_i} = \bar{x} \underset{\sum p_i = 1}{\Rightarrow} p_i = \frac{1}{d}$ $\forall i$.

- DEF. (JOINT PROB. DISTRIBUTION $P(X,Y)$)
  - $X \equiv \{x_1, \dots, x_d\}$, $Y \equiv \{y_1, \dots, y_D\}$

  $P(x,y)$ joint prob. distribution $\overset{\text{DEF.}}{\Longleftrightarrow}$ $\begin{cases} \cdot\ P(x,y) \geq 0 \quad \forall x \in X, y \in Y \\ \cdot\ \sum_{\substack{x\in X \\ y \in Y}} P(x,y) = 1 \end{cases}$

- DEF. (JOINT ENTROPY $H(X,Y)$)

$$H(X,Y) = -\sum_{\substack{x\in X \\ y\in Y}} P(x,y)\log(P(x,y))$$

DEF. (MARGINAL DISTRIBUTION $P(x)$ and $P(y)$ given $P(x,y)$).

- Given JOINT PROB. DISTRIB. $P(x,y)$, we define:

$$P(x) := \sum_{y\in Y} P(x,y)$$ (marginal distrib. of X.)

$$P(y) := \sum_{x \in X} P(x,y)$$ (marginal distrib. of Y.)

($\cdot\ P(x) \geq 0$, $\sum_x P(x) = 1$
$\cdot\ H(X) = -\sum_{x\in X} P(x)\log(P(x))$)

($\cdot\ P(y) \geq 0$, $\sum_y P(y) = 1$
$H(Y) = -\sum_{y\in Y} P(y)\log(P(y))$)

• TH 1 (SUB-ADDITIVE PROPERTY OF $H(X,Y)$)

$$H(X,Y) \leq H(X) + H(Y) \qquad \left(H(X,Y) = H(X) + H(Y) \iff P(x,y) \underset{\substack{\uparrow \\ \text{INDIP. RANDOM VARIABLES}}}{=} P(x)\cdot P(y) \right)$$

PROOF:

$$H(x,y) - H(x) - H(y) = -\sum_{x,y} P(x,y) \log(P(x,y)) + \sum_{x} P(x) \log(P(x)) + \sum_{y} P(y) \log(P(y))$$

$$= -\sum_{x,y} P(x,y) \log(P(x,y)) + \sum_{x,y} P(x,y) \log(P(x)) + \sum_{x,y} P(x,y) \log(P(y))$$

- $P(x) = \sum_{y} P(x,y)$
- $P(y) = \sum_{x} P(x,y)$

$$= -\sum_{x,y} P(x,y) \log\left(\frac{P(x,y)}{P(x)\,P(y)}\right) = \sum_{x,y} P(x,y) \log\left(\frac{P(x)\,P(y)}{P(x,y)}\right)$$

$$\leq \log\left(\sum_{x,y} \cancel{P(x,y)} \frac{P(x)\,P(y)}{\cancel{P(x,y)}}\right) = \log(1) = 0$$

(• $\log(\cdot)$ CONCAVE)

- $\sum_{x} P(x) = 1$
- $\sum_{y} P(y) = 1$

• Since $\log(\cdot)$ is strictly CONCAVE $\Longrightarrow$
the inequality is saturated when $\frac{P(x)\,P(y)}{P(x,y)} = \text{CONST} \;\; \forall x,y.$

$\iff P(x,y) = P(x)\cdot P(y)$.

Product distribution.

TH 2

$$H(X) \leq H(X,Y) \qquad \left(\text{"="} \iff P(X,Y) = P(X)\, \delta_{Y,\bar{Y}_0} \right)$$

$$H(Y) \leq H(X,Y) \qquad \left(\text{"="} \iff P(X,Y) = P(Y)\, \delta_{X,\bar{X}_0} \right)$$

$$\left(\text{or} \quad \max\big(H(X), H(Y)\big) \leq H(X,Y) \right)$$

PROOF:

$$H(X) - H(X,Y) = -\sum_{x} \underbrace{P(x)}_{=\sum_y P(x,y)} \log\big(P(x)\big) + \sum_{x,y} P(x,y) \log\big(P(x,y)\big)$$

$$= -\sum_{x,y} P(x,y) \log\left(\frac{P(x)}{P(x,y)}\right)$$

$$= \sum_{x,y} P(x,y) \log\left(\frac{P(x,y)}{P(x)}\right) \underset{\log(\cdot)\ \text{CONCAVE}}{\leq} \log\left(\sum_{x,y} \frac{P^2(x,y)}{P(x)}\right)$$

$$\leq \log\left(\sum_{x,y} \frac{P^2(x,y)}{P(x)}\right) \leq \log\left(\sum_{x} \frac{\cancel{P^2(x)}}{\cancel{P(x)}}\right) = \log(1) = 0 \implies \square$$

- $\dfrac{P^2(x,y)}{P(x)} = \dfrac{P(x,y)}{P(x)} \cdot P(x,y) \leq 1 \cdot P(x,y)$
  - $P(x) = \sum_{y'} P(x,y') \Rightarrow \dfrac{P(x,y)}{P(x)} \leq 1$
  - SATURATED $\iff \dfrac{P(x,y)}{P(x)} = 1 \iff P(x,y) = P(x) \cdot \delta_{y,\bar{y}_0}$
- $\log(\cdot)$ monotone

- We'll see that $\max\big(H(X), H(Y)\big) \leq H(X,Y)$ is violated in the "QUANTUM VERSION" due to ENTANGLEMENT.

DEF: (CONDITIONAL PROBABILITY $P(X|Y)$)

$$P(x|y) := \frac{P(x,y)}{P(y)}$$

(Fixed y, $P(x|y)$ is a prob. dist.: $\begin{cases} \cdot\ P(x|y) \geq 0 \\ \cdot\ \sum_x P(x|y) = 1 \end{cases}$)

$\sum_x P(x,y) = P(y)$

This defines a random variable $X|_y$ distributed according to $P(x|y)$.

$$P(y|x) := \frac{P(x,y)}{P(x)}$$

(Fixed x, $P(y|x)$ is a prob. dist.: $\begin{cases} \cdot\ P(y|x) \geq 0 \\ \cdot\ \sum_{\cancel{x}} P(y|x) = 1 \end{cases}$)

This defines a random variable $Y|_x$

OBS! We have $\sum_{\cancel{x}} P(x|y)\,P(y) = P(x)$

DEF: (CONDITIONAL ENTROPY $H(X|Y)$)

$$H(X|Y) := \mathbb{E}_Y\, H(X|_y) = \mathbb{E}_Y\left(-\sum_x P(x|y)\log\left(P(x|y)\right)\right)$$

COR:

$$H(X|Y) = -\sum_{x,y} P(x,y)\log\left(P(x|y)\right)$$
$$= H(X,Y) - H(Y)$$

PROOF:

- $$H(X|Y) = \mathbb{E}_Y\left(-\sum_x P(x|y)\log\left(P(x|y)\right)\right)$$
$$= -\sum_{x,y} P(x|y)\,P(y)\log\left(P(x|y)\right) =$$
$$\underset{P(x|y) := \frac{P(x,y)}{P(y)}}{=} -\sum_{x,y} P(x,y)\log\left(P(x|y)\right)$$

- $$H(X|Y) = -\sum_{x,y} P(x,y)\log\left(P(x|y)\right) = -\sum_{x,y} P(x,y)\log\left(\frac{P(x,y)}{P(y)}\right) =$$
$$= -\sum_{x,y} P(x,y)\log\left(P(x,y)\right) + \sum_{y,\cancel{x}} \underbrace{P(x,y)}_{P(y)}\log\left(P(y)\right) = H(X,Y) - H(Y)$$

OBS.

$$H(X|Y) \geq 0$$

PROOF:

$$H(X|Y) = H(X,Y) - H(Y) \geq 0$$

THEOREM 2

(This inequality does not hold for the Quantum version of $H(X|Y)$ because TH. 2 is violated.)

DEF. (MUTUAL INFORMATION)

$$I(X:Y) := H(X) + H(Y) - H(X,Y)$$

OBS.

- $I(X:Y) \geq 0$

  (SUB-ADD. (TH. 2))

- $I(X:Y) = I(Y:X)$

  $H(X,Y) = H(Y,X)$

- $I(X:Y) = H(X,Y) - H(X|Y) - H(Y|X)$

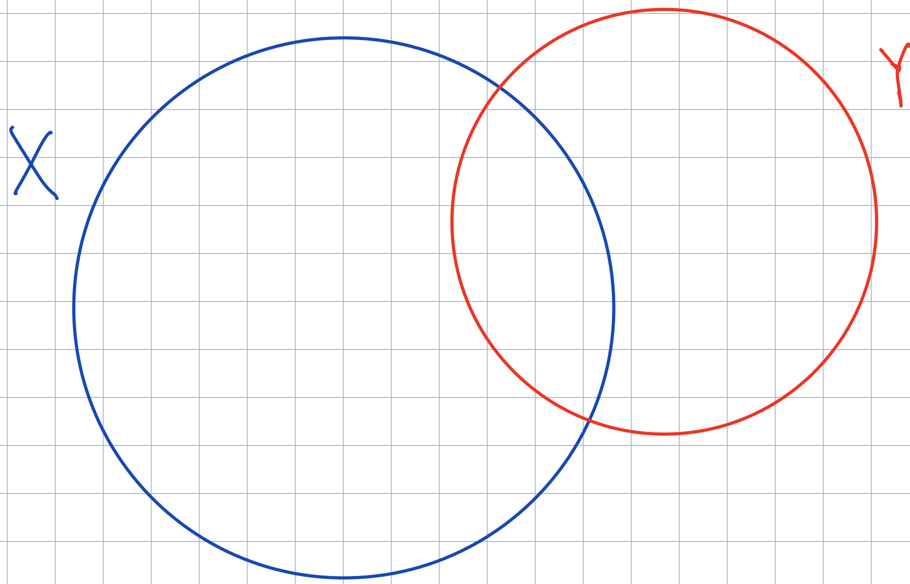
  $I(X:Y) = \left(H(X,Y) - H(X|Y)\right) + \left(H(X,Y) - H(Y|X)\right) - H(X,Y)$

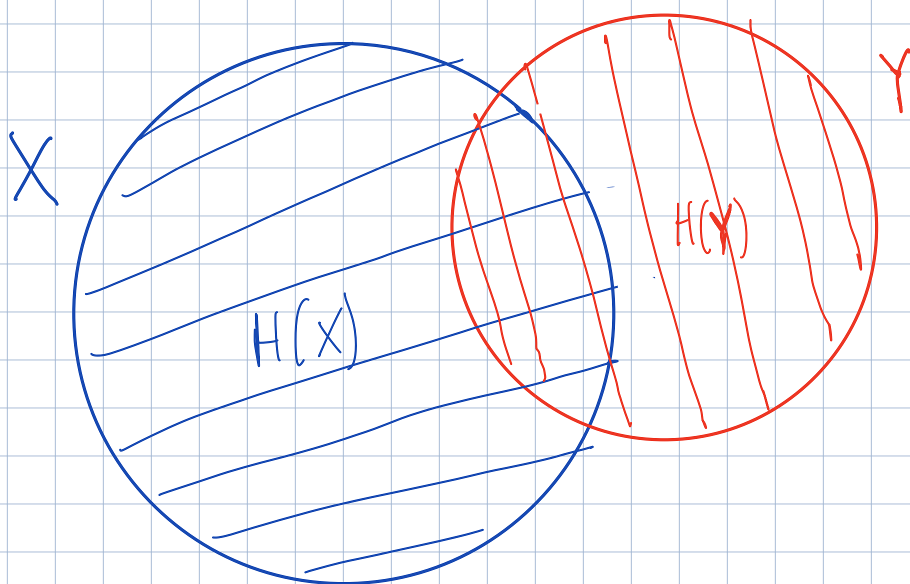
  - $H(X|Y) = H(X,Y) - H(Y)$
  - $H(Y|X) = H(X,Y) - H(X)$

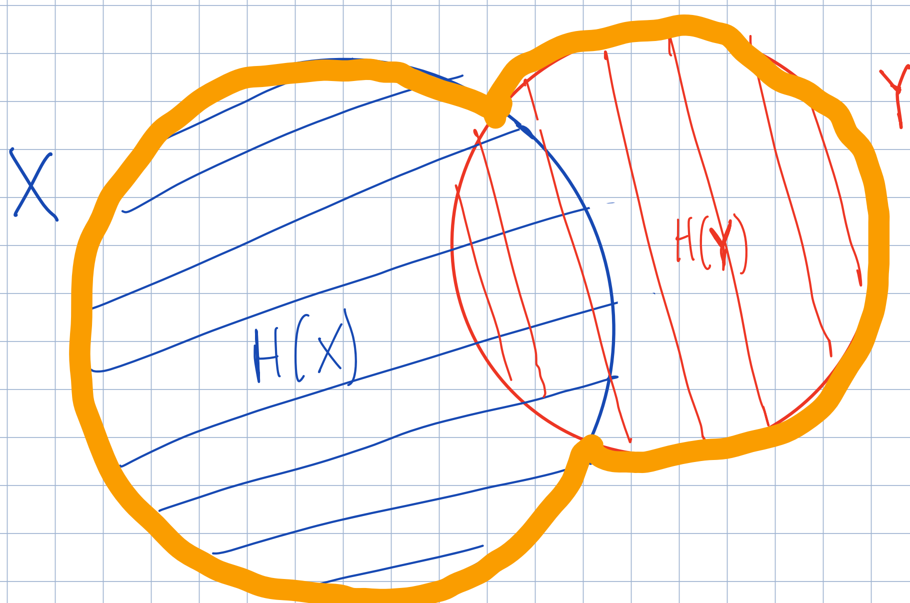
- $I(X:Y) \leq \min\left(H(X), H(Y)\right)$

  $I(X:Y) = H(X) + H(Y) - H(X,Y) \leq H(X) + H(Y) - \max\left(H(X), H(Y)\right) = \min\left(H(X), H(Y)\right)$

GRAPHICAL INTERPRETATION

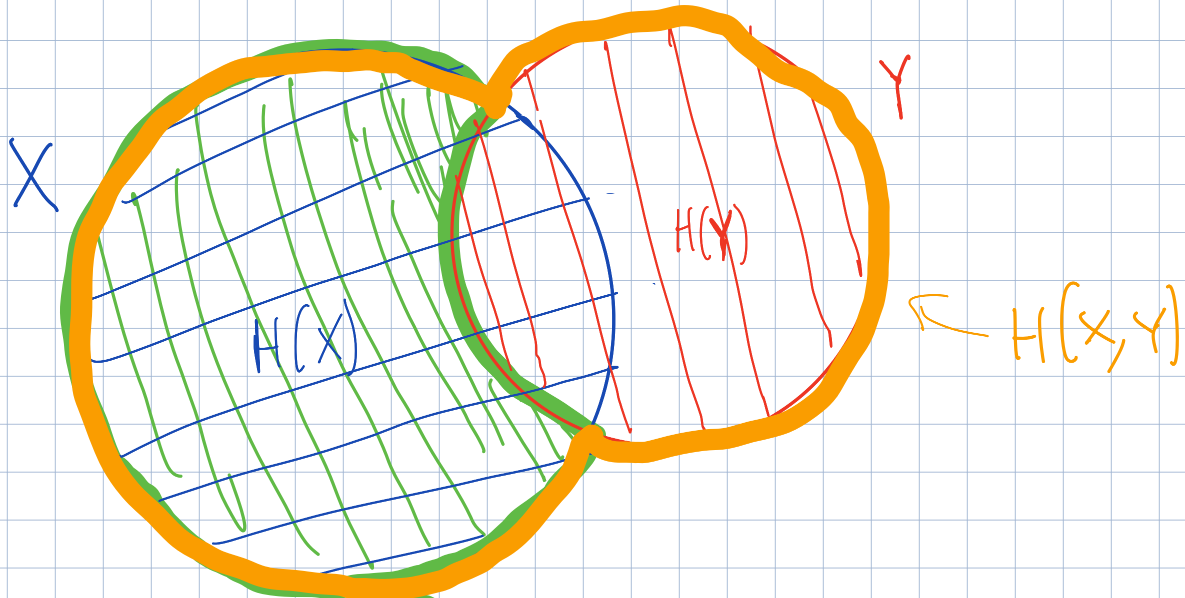




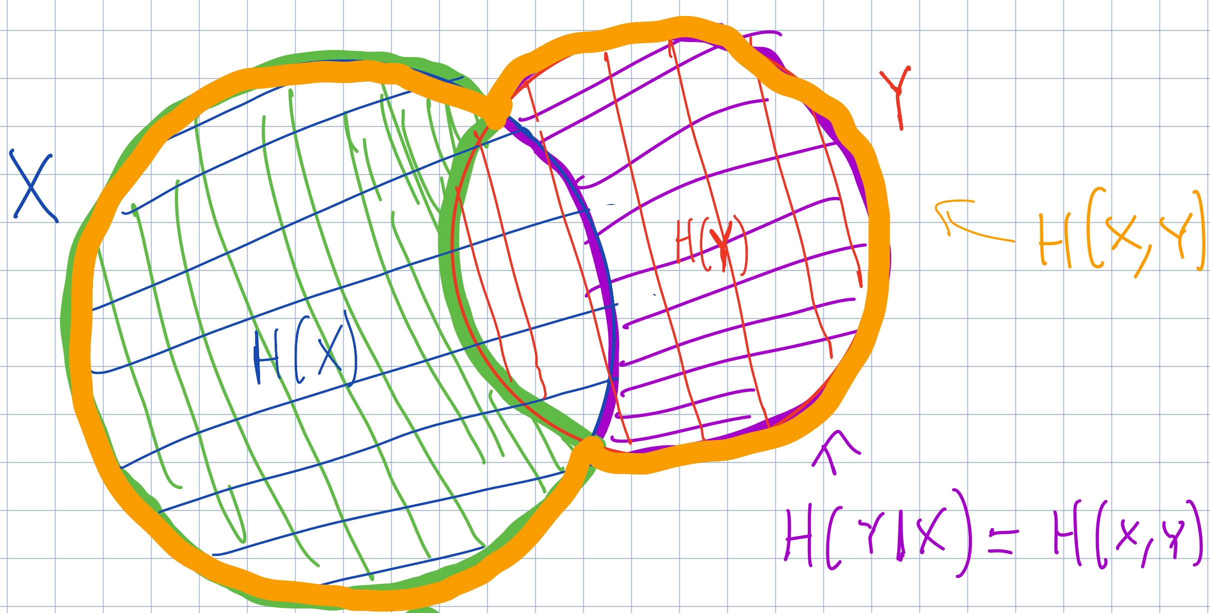


$H(X,Y)$

- $H(X,Y) \leq H(X) + H(Y)$
- $H(X,Y) \geq \max(H(X,Y))$
- They are obvious here!

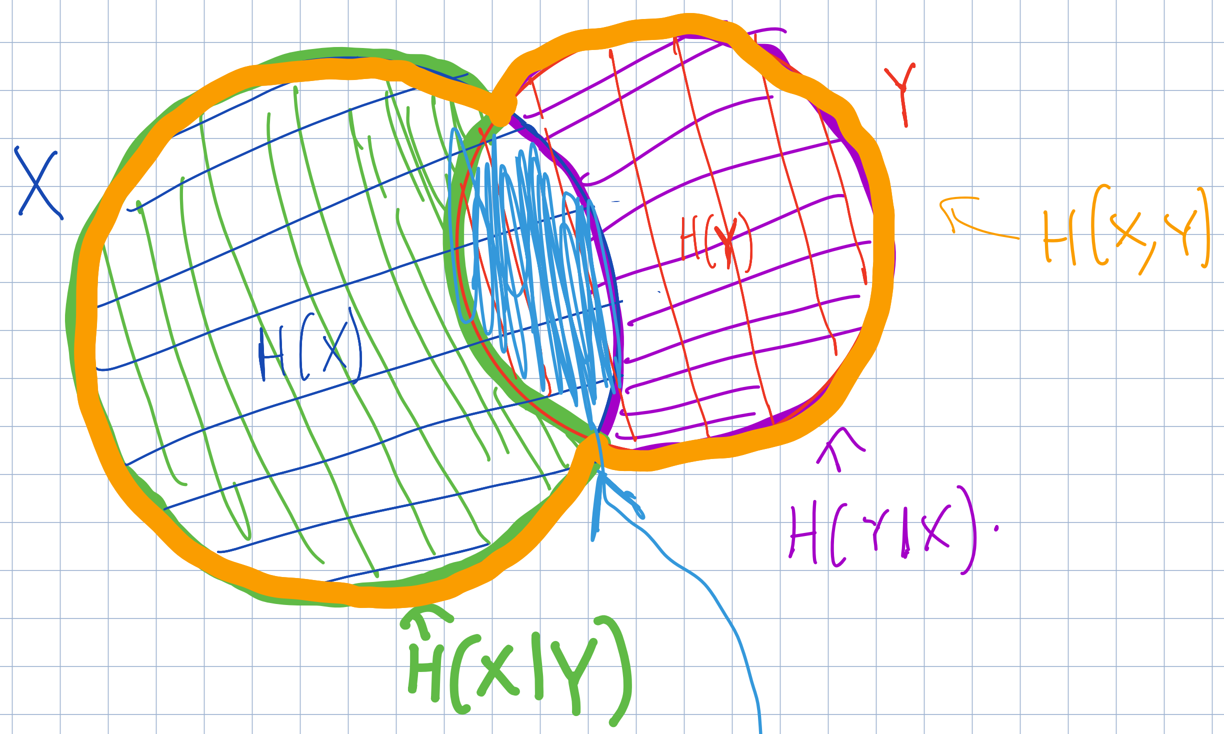


$H(X|Y) = H(X,Y) - H(Y)$



$H(Y|X) = H(X,Y) - H(X)$

$H(X|Y)$



$$I(X:Y) = H(X,Y) - H(X|Y) - H(Y|X)$$

- DEF: (RELATIVE ENTROPY)

- $p(x) \geq 0$; $\sum_x p(x) = 1$. $q(x) \geq 0$; $\sum_x q(x) = 1$.

$P(x = P(x,Y)$
$q(x) = x) \cdot P(y$

$$H\left(p(x) \,\|\, q(x)\right) := \sum_x p(x) \log_2\left(\frac{p(x)}{q(x)}\right) =$$

$$= -H(p) - \sum_x p(x) \log_2(q(x))$$

• OBS. $H\left(p(x) \| q(x)\right) \geq 0$ $\left(H\left(p(x) \| q(x)\right) = 0 \iff p(x) = q(x)\ \forall x \right)$

PROOF:

$$H\left(p(x) \| q(x)\right) := \sum_x p(x) \log_2\left(\frac{p(x)}{q(x)}\right) \geq \sum_x \frac{p(x)}{\ln(2)}\left(1 - \frac{q(x)}{p(x)}\right)$$

• $e^x \geq 1 + x$

$\Rightarrow x \geq \ln(1+x)$

$\Rightarrow y - 1 \geq \ln(y) = \log_2(y)\ln(2)$

$\Rightarrow -\log_2(y) \geq \frac{1-y}{\ln(2)}$

• $\log_2\left(\frac{p}{q}\right) = -\log_2\left(\frac{q}{p}\right)$

$$= \frac{1}{\ln(2)}(1 - 1) = 0$$

• The inequality $\log_2(y) \geq \frac{1}{\ln(2)}(1 - y)$ is saturated $\iff y = 1 \iff p(x) = q(x)$

OBS:

• $H\left(p(x,y) \| p(x) \cdot p(y)\right) \underset{\text{DEF}}{=} -H(X,Y) + H(X) + H(Y) \underset{\text{OBS.}}{\geq} 0$

$$\Rightarrow H(X,Y) \leq H(X) + H(Y)$$

• $H\left(p(x) \| q(x)\right) \underset{q(x) = \frac{1}{d}\ \forall x}{=} -H(X) - \underbrace{\sum_x p(x)}_{1} \log\left(\frac{1}{d}\right) \underset{\text{OBS}}{\geq} 0 \Rightarrow H(X) \leq \log(d)$

2. **On Shannon entropy...**

To begin with let us first show some simple properties of entropies, in particular, of the mutual information.

Recall the definition of the Shannon entropies for random variables X, Y which take values in $\mathcal{X}, \mathcal{Y}$ and are distributed according to probability distributions p, q over $\mathcal{X}$ and $\mathcal{Y}$, respectively.

$$H(X) = -\sum_{x \in \mathcal{X}} p(x) \log p(x) \text{ (Shannon entropy)} \tag{1}$$

$$H(X|Y) = H(X,Y) - H(Y) = \sum_{x \in \mathcal{X}} p(x) H(Y|X = x) \text{ (Conditional entropy)} \tag{2}$$

a) Show that $0 \leq H(X) \leq \log |\mathcal{X}|$, where the first equality holds *iff* there is an $x \in \mathcal{X}$ for which $p(x) = 1$ and the second inequality holds *iff* $p(x) = 1/|\mathcal{X}|$ for all x.

b) Show that the Shannon entropy is *subadditive*, i.e., that $H(X,Y) \leq H(X) + H(Y)$ with equality if X and Y are independent..

Hint: Show that $H(X,Y) - H(X) - H(Y) \leq 0$ *using that* $\log_2 x \ln 2 = \ln x \leq x - 1$.

c) Show that $H(Y|X) \geq 0$ equality if and only if Y is a (deterministic) function of X.

Hint: Use Bayes' rule: $p(x,y) = p(y|x)p(x)$

• We showed all these points explaining the theory.

• DEF. VON NEUMAN ENTROPY

$$S(\rho) := -\mathrm{tr}(\rho \log \rho) \quad \text{where } \rho \text{ is a density matrix}$$

• $\rho = \sum_{i=1}^{d} p_i |v_i\rangle\langle v_i| \Longrightarrow S(\rho) = -\sum_{i=1}^{d} p_i \log(p_i) = H(\{p_i\})$

EIGENDEC.

eigenvalues (with $\sum_{i=1}^{d} p_i = 1$)

SHANNON ENTROPY

PROOF:

In general we have $f(\rho) = \sum_{i=1}^{d} f(p_i) |v_i\rangle\langle v_i|$.

FACT (★):

This because every $f(\cdot)$ can be written as $f(\rho) = \sum_{j=0}^{\infty} c_j \rho^j$

$$\Rightarrow f(\rho) = \sum_{j=0}^{\infty} c_j \rho^j \underbrace{\mathbb{1}}_{\sum_{i=1}^{d} |v_i\rangle\langle v_i|} = \sum_i \sum_j c_j \underbrace{(\rho^j |v_i\rangle)}_{p_i^j |v_i\rangle} \langle v_i| =$$

$$= \sum_i \left(\sum_j c_j p_i^j \right) |v_i\rangle\langle v_i| = \sum_{i=1}^{d} f(p_i) |v_i\rangle\langle v_i|$$

$\sum_j c_j p_i^j = f(p_i)$

• So given $f(\rho) := \rho \log(\rho) \Rightarrow \rho \log(\rho) = \sum_{i=1}^{d} (p_i \log(p_i)) |v_i\rangle\langle v_i|$

$$\Rightarrow S(\rho) := -\mathrm{tr}(\rho \log \rho) = -\sum_{i=1}^{d} (p_i \log(p_i)) \underbrace{\mathrm{tr}(|v_i\rangle\langle v_i|)}_{1}$$

$$= -\sum_{i=1}^{d} p_i \log p_i$$

OBS.

$$0 \leq S(\rho) \leq \log(d)$$

PROOF:

$S(\rho) = H(\{p_i\})$ and we showed that $0 \leq H(X) \leq \log(d)$.

OBS |

- $S(\rho) = 0 \iff \rho = |\psi\rangle\langle\psi|$ (PURE)

- $S(\rho) = \log(d) \iff \rho = \frac{\mathbb{1}}{d}$ (MAXIMALLY MIXED STATE)

PROOF:

- $S(\rho) = H(\{p_i\}) = 0 \underset{\substack{\uparrow \\ \text{PREVIOUS LECTURE}}}{\iff} \{p_i\}$ is a peaked distribution i.e. $p_i = \delta_{i,a}$ $\forall i$.

- $S(\rho) = H(\{p_i\}) = \log(d) \iff \{p_i\}$ is a flat distribution i.e. $p_i = \frac{1}{d}$ $\forall i$.

$$\iff \rho = \sum_{i=1}^{d} p_i |v_i\rangle\langle v_i| = \frac{1}{d} \underbrace{\sum_{i=1}^{d} |v_i\rangle\langle v_i|}_{\mathbb{1}} \Rightarrow \square$$

- $S(\rho) = S(U\rho U^\dagger)$ $\forall\, U$ unitary.

PROOF:

$\rho = \sum_i p_i |v_i\rangle\langle v_i|$ and $U\rho U^\dagger = \sum_i p_i (U|v_i\rangle)(\langle v_i| U^\dagger)$ have the same eigenvalues.

and $S(\cdot)$ depends only by the eigenvalues.

- DEF: (RELATIVE ENTROPY)

$$S(\rho_1 \| \rho_2) := \mathrm{tr}[\rho_1 \log(\rho_1)] - \mathrm{tr}[\rho_1 \log(\rho_2)]$$

$$= -S(\rho_1) - \mathrm{tr}[\rho_1 \log(\rho_2)]$$

TEO) (1): (KLEIN INEQUALITY)

$$S(\rho_1 \| \rho_2) \geq 0 \qquad \left(S(\rho_1 \| \rho_2) = 0 \iff \rho_1 = \rho_2 \right)$$

PROOF:

$$\rho_1 = \sum_{i=1}^{d} p_i |v_i\rangle\langle v_i| \quad , \quad \rho_2 = \sum_{j=1}^{d} q_j |w_j\rangle\langle w_j|$$

$$S(\rho_1 \| \rho_2) = -S(\rho_1) - tr\left[\rho_1 \log(\rho_2)\right]$$

$$= -S(\rho_1) - tr\left[\sum_{i=1}^{d} p_i |v_i\rangle\langle v_i| \log(\rho_2)\right]$$

$$= -S(\rho_1) - \sum_{i=1}^{d} p_i \langle v_i| \log(\rho_2) |v_i\rangle$$

$$= -S(\rho_1) - \sum_{i=1}^{d} \sum_{j=1}^{d} p_i \log(q_j) \underbrace{\langle v_i|w_j\rangle\langle w_j|v_i\rangle}_{= |\langle v_i|w_j\rangle|^2} =$$

$$\log(\rho_2) = \sum_{j=1}^{d} \log(q_j) |w_j\rangle\langle w_j|$$

(*)

$$= -S(\rho_1) - \sum_{i=1}^{d} \sum_{j=1}^{d} p_i \log(q_j) |\langle v_i|w_j\rangle|^2$$

$$= +\sum_{i=1}^{d} p_i \log(p_i) - \sum_{i=1}^{d} \sum_{j=1}^{d} p_i \log(q_j) |\langle v_i|w_j\rangle|^2$$

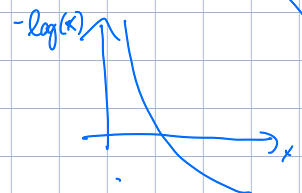
$$= +\sum_i \sum_j |\langle v_i|w_j\rangle|^2 p_i \log(p_i) - \sum_{i=1}^{d} \sum_j |\langle v_i|w_j\rangle|^2 p_i \log(q_j)$$

$$\sum_j |\langle v_i|w_j\rangle|^2 = 1$$

$$= +\sum_i \sum_j |\langle v_i|w_j\rangle|^2 p_i \left(\log\left(\frac{p_i}{q_j}\right)\right) = \sum_i \sum_j \underbrace{|\langle v_i|w_j\rangle|^2 p_i}_{=: P(i,j)} \left(-\log\left(\frac{q_j}{p_i}\right)\right)$$

$$\geq \sum_i p_i \left[-\log\left(\sum_j P(i,j)\, \frac{q_j}{p_i}\right)\right] = \sum_i p_i \left[-\log\left(\frac{r_i}{p_i}\right)\right]$$

- $\sum_j P(i,j) = 1$, $P(i,j) \geq 0$
- $-\log(x)$ CONVEX



- $r_i := \sum_j P(i,j)\, q_j$ } Prob. dist.
  - $\sum_i r_i = 1$
  - $r_i \geq 0$

$$= \sum_i p_i \log\left(\frac{p_i}{r_i}\right) \geq 0$$

previous lecture

- The first inequality is saturated when $p_i = q_j \ \forall i,j$

  $\Updownarrow$

  $p_i = q_j = \frac{1}{d} \ \forall i,j$

  or when $P(i,j) = \delta_{j,i_0}$. $\forall j = 1,\dots,d$

- But $P(i,j) = |\langle V_i|W_j\rangle|^2 = \delta_{j,i_0} \ \forall j = 1,\dots,d \iff |V_i\rangle = |W_{i_0}\rangle \ \forall i$

  $|V_i\rangle = \sum_j \underbrace{\langle W_j|V_i\rangle}_{=\,\delta_{j,i_0} e^{i\phi(i_0)}} |W_j\rangle$

  $= e^{i\phi(i_0)} |W_{i_0}\rangle$

- The second inequality is saturated when $p_i = r_i \ \forall i$ (previous lecture)

  $\iff p_i = \sum_j P(i,j)\, q_j = \sum_j \delta_{j,i_0}\, q_j = q_{i_0} \quad \forall i$

  $\iff \rho_1 = \sum_i p_i |V_i\rangle\langle V_i| = \sum_i q_{i_0} |W_{i_0}\rangle\langle W_{i_0}| = \rho_2$

• OBS.

• $S(\rho_1 \| \rho_2) \neq S(\rho_2 \| \rho_1)$ (NOT SYMMETRIC)

• $\mathrm{Supp}(\rho_1) \cap \mathrm{Ker}(\rho_2) \neq \{\vec{0}\} \Rightarrow S(\rho_1 \| \rho_2) = +\infty$

PROOF:

$$S(\rho_1 \| \rho_2) = -S(\rho_1) - \sum_{i=1}^{d} \sum_{j=1}^{d} p_i \log(q_j) |\langle v_i | w_j \rangle|^2$$

If $q_j = 0$ that $p_i |\langle v_i | w_j \rangle| \neq 0 \ \forall i \Longrightarrow S(\rho_1 \| \rho_2) = +\infty$

$\Updownarrow$ $|w_j\rangle \in \mathrm{Ker}(\rho_2)$ $\quad$ $\Updownarrow$ $|w_j\rangle \notin \mathrm{Ker}^{\perp}(\rho_1) = \mathrm{supp}(\rho_1)$

• So $\mathrm{Ker}(\rho_2) \cap \mathrm{supp}(\rho_1) \neq \{0\} \Rightarrow S(\rho_1 \| \rho_2) = +\infty \Rightarrow \square$

• $O = O_1 \otimes O_2$ with $O_1 = O_1^{\dagger}$, $O_2 = O_2^{\dagger}$

$$\log(O) = \log(O_1) \otimes \mathbb{1} + \mathbb{1} \otimes \log(O_2) = \star$$

PROOF:

$$\log(O) = \log(O_1 \otimes O_2) \underset{\uparrow}{=} \log\left[\sum_{i,j} \lambda_i^{(1)} \lambda_j^{(2)} \left(|v_i^{(1)}\rangle\langle v_i^{(1)}| \otimes |v_j^{(2)}\rangle\langle v_j^{(2)}|\right)\right] =$$

$O_1 = \sum_i \lambda_i^{(1)} |v_i^{(1)}\rangle\langle v_i^{(1)}|$

$O_2 = \sum_j \lambda_j^{(2)} |v_j^{(2)}\rangle\langle v_j^{(2)}|$

$\uparrow$ eigenvectors for $O_1 \otimes O_2$

$$\underset{\uparrow}{=} \sum_{i,j} \log(\lambda_i^{(1)} \lambda_j^{(2)}) \left(|v_i^{(1)}\rangle\langle v_i^{(1)}| \otimes |v_j^{(2)}\rangle\langle v_j^{(2)}|\right) \underset{\uparrow}{=}$$

$f(A) = \sum_i f(\lambda_i) |v_i\rangle\langle v_i|$, $A = \sum_i \lambda_i |v_i\rangle\langle v_i|$

$\log(ab) = \log(a) + \log(b)$

$$= \sum_{i,j} \log(\lambda_i^{(1)}) \left(|v_i^{(1)}\rangle\langle v_i^{(1)}| \otimes |v_j^{(2)}\rangle\langle v_j^{(2)}|\right) + \sum_{i,j} \log(\lambda_j^{(2)}) \left(|v_i^{(1)}\rangle\langle v_i^{(1)}| \otimes |v_j^{(2)}\rangle\langle v_j^{(2)}|\right)$$

$$= \sum_i \log(\lambda_i^{(1)}) \left(|v_i^{(1)}\rangle\langle v_i^{(1)}| \otimes \mathbb{1}\right) + \sum_j \log(\lambda_j^{(2)}) \left(\mathbb{1} \otimes |v_j^{(2)}\rangle\langle v_j^{(2)}|\right) =$$

$$= \log(O_1) \otimes \mathbb{1} + \log(O_2) \otimes \mathbb{1}$$

COR.

- $O = O_1 \otimes O_2 \otimes \cdots \otimes O_n \quad , O_i = O_i^\dagger \quad \forall i$
- $\log(O) = \sum_{i=1}^{N} \mathbb{1}_{1,\dots,i-1} \otimes \log(O_i) \otimes \mathbb{1}_{i+1,\dots,n}$

COR:

$$S(\rho_1 \otimes \rho_2 \otimes \dots \otimes \rho_N) = \sum_{i=1}^{N} S(\rho_i)$$

PROOF:

$$\log(\rho) = \sum_{i=1}^{N} \mathbb{1}_{1,\dots,i-1} \otimes \log(\rho_i) \otimes \mathbb{1}_{i+1,\dots,n}$$

$$S(\rho) = -\mathrm{tr}\left(\rho \log(\rho)\right) = -\sum_{i=1}^{N} \mathrm{tr}\left(\rho \left(\mathbb{1}_{1,\dots,i-1} \otimes \log(\rho_i) \otimes \mathbb{1}_{i+1,\dots,n}\right)\right) =$$

$$= -\sum_{i=1}^{N} \mathrm{tr}\Big(\underbrace{\mathrm{tr}_{\bar{i}}(\rho)}_{=\rho_i} \log(\rho_i)\Big) = \sum_{i=1}^{N} S(\rho_i)$$

THEOR. 2

- $\rho_A = \mathrm{tr}_B(\rho_{AB}) \quad , \rho_B = \mathrm{tr}_A(\rho_{AB})$
- $S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B) \quad \left(S(\rho_{AB}) = S(\rho_A) + S(\rho_B) \Leftrightarrow \rho_{AB} = \rho_A \otimes \rho_B\right)$

PROOF:

- $S(\rho_{AB} \| \rho_A \otimes \rho_B) \geq 0$

$\uparrow$ KLEIN INEQUALITY

- $\underbrace{S(\rho_{AB} \| \rho_A \otimes \rho_B)}_{\geq 0} = -S(\rho_{AB}) - \mathrm{tr}\Big(\rho_{AB} \underbrace{\log(\rho_A \otimes \rho_B)}_{= \log(\rho_A) \otimes \mathbb{1} + \mathbb{1} \otimes \log(\rho_B)}\Big)$

$$= -S(\rho_{AB}) - \underbrace{\mathrm{tr}\left(\rho_{AB} \log(\rho_A)\otimes \mathbb{1}\right)}_{\mathrm{tr}(\rho_A \log(\rho_A))} - \mathrm{tr}\left(\rho_{AB}\, \mathbb{1}\otimes \log(\rho_B)\right)$$

$$= -S(\rho_{AB}) + S(\rho_A) + S(\rho_B) \geq 0 \implies \blacksquare$$

- "=" $\iff S(\rho_{AB} \| \rho_A \otimes \rho_B) = 0 \iff \rho_{AB} = \rho_A \otimes \rho_B$

FACT ③:

- $\rho_A := \mathrm{tr}_B(\rho)$, $\rho_B := \mathrm{tr}_A(\rho)$
- $\rho = |\psi\rangle\langle\psi|$ pure $\implies S(\rho_A) = S(\rho_B)$

(This is not true in general if ρ is not pure)
Find a counter-example ☺.

PROOF:

SCHMIDT.

$$|\psi\rangle = \sum_{i=1}^{\min(d_A,d_B)} \sqrt{\lambda_i}\, |v^A_i\rangle \otimes |v^B_i\rangle \quad \text{with } \langle v^A_i, v^A_j\rangle = \delta_{i,j},\ \langle v^B_i, v^B_j\rangle = \delta_{i,j}.$$

SUBPROOF:

- $$|\psi\rangle = \sum_{i,j} \underbrace{\langle i,j|\psi\rangle}_{(C)_{i,j}} |i,j\rangle \underset{\text{SVD}}{=} \sum_{i,j} (UDV^{\dagger})_{i,j} |i,j\rangle =$$

$\bullet\ C = UDV^{\dagger}$

$$= \sum_{\ell}\sum_{i,j} U_{i,\ell} D_{\ell\ell} (V^{\dagger})_{\ell,j} |i\rangle\otimes|j\rangle = \sum_{\ell} \lambda_\ell \left(\underbrace{\sum_i U_{i\ell}|i\rangle}_{U|\ell\rangle = |v_i^A\rangle}\right) \otimes \left[\underbrace{\sum_j (V^{\dagger})_{\ell,j}|j\rangle}_{V^*|\ell\rangle = |v_i^B\rangle}\right]$$

$\langle i|U|\ell\rangle = U_{i\ell}$ $\qquad$ $\sqrt{\lambda_\ell} := D_{\ell\ell} \geq 0$

$$\mathrm{tr}_B(|\psi\rangle\langle\psi|) = \sum_i \lambda_i\, |v_i^{(A)}\rangle\langle v_i^{(A)}|$$

$$\mathrm{tr}_A(|\psi\rangle\langle\psi|) = \sum_i \lambda_i\, |v_i^{(B)}\rangle\langle v_i^{(B)}|$$

$\implies$ Same eigenvalues $\implies$ same entropies.

THEOREM ④

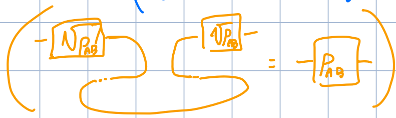
$$S(\rho_{AB}) \geq |S(\rho_A) - S(\rho_B)|$$

PROOF:

- Given a mixed state ρ_{AB}, then $\exists$ R such that $\rho_{AB} = tr_R[\underbrace{|\Psi\rangle\langle\Psi|}_{\overset{!!}{\rho_{ABR}}}]$ with $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_R$

SUBPROOF: (ALREADY SEEN)

(E.g. take $|\Psi\rangle := \sqrt{\rho_{AB}} \otimes \mathbb{1}_R \left(\sum_{i=1}^{d} |i\rangle_{AB} \otimes |i\rangle_R \right)$ and verify.



)

- We have:

- $S(\rho_B) \underset{\uparrow}{=} S(\rho_{AR}) \underset{\substack{\uparrow \\ (\text{TH}②)}}{\leq} S(\rho_A) + S(\rho_R) \underset{\uparrow}{=} S(\rho_A) + S(\rho_{AB})$

[$\cdot\ S(\rho_B) \underset{\uparrow}{=} S(\rho_{AR})$ (· FACT ③ · ρ_{ABR} pure)]

[$\cdot\ S(\rho_{AB}) \underset{\uparrow}{=} S(\rho_R)$ (· FACT ③ · ρ_{ABR} pure)]

$$\Longrightarrow \boxed{S(\rho_{AB}) \geq S(\rho_B) - S(\rho_A)} \equiv \underline{*_1}$$

- $S(\rho_A) \underset{\substack{\uparrow \\ \text{FACT}③}}{=} S(\rho_{BR}) \underset{\substack{\uparrow \\ \text{TH}②}}{\leq} S(\rho_B) + S(\rho_R) \underset{\substack{\uparrow \\ \text{FACT}③}}{=} S(\rho_B) + S(\rho_{AB})$

$$\Longrightarrow \boxed{S(\rho_{AB}) \geq S(\rho_A) - S(\rho_B)} \equiv \underline{*_2}$$

$\underset{\substack{\uparrow \\ *_1,\ *_2}}{\Longrightarrow}$ $\square$

OBS | Also in the classical case I have $H(X,Y) \geq |H(X) - H(Y)|$ but we did not mention it because in the classical case we have an even stronger inequality: $H(X,Y) \geq \max(H(X), H(Y))$ which can be violated quantumly.

$$\boxed{S(\rho_{AB}) \geq \max(S(\rho_A), S(\rho_B))}$$

FALSE

NO!

FALSE!

NO!

NO!

PROOF.

COUNTER-EXAMPLE: $\rho_{AB} = |\psi\rangle\langle\psi|$ with $|\psi\rangle = |\Omega\rangle$ max. entangled state.

- $S(\rho_{AB}) = 0$ ← state pure.
- $\max(S(\rho_A), S(\rho_B)) = \max\left(\underbrace{S\left(\frac{\mathbb{1}}{d_A}\right)}_{\neq 0}, \underbrace{S\left(\frac{\mathbb{1}}{d_B}\right)}_{\neq 0}\right) \neq 0$

DEF|.

$$\rho \text{ separable} \iff \rho = \sum_{i=1}^{N} p_i\, \rho_1^{(i)} \otimes \rho_2^{(i)} \quad \text{with } \begin{cases} p_i \geq 0 \\ \sum_{i=1}^{N} p_i = 1 \end{cases}$$

$\rho_1^{(i)}$ and $\rho_2^{(i)}$ states.

DEF|

$$\rho \text{ entangled} \iff \rho \text{ Not separable.}$$

FACT 1:

$$\rho \text{ separable} \implies S(\rho_{AB}) \geq S(\rho_A)$$

PROOF: NOT SHOWN HERE.

$\boxed{\text{THEOR.}}$ (CONCAVITY)

$$S\left(\sum_{i=1}^{N} p_i \rho_i\right) \geq \sum_{i=1}^{N} p_i S(\rho_i) \qquad \left(S\left(\sum_{i=1}^{N} p_i \rho_i\right) = \sum_{i=1}^{N} p_i S(\rho_i) \iff \rho_i = \rho \ \forall i \right)$$

PROOF:

- $$\rho_{AB} := \sum_{i=1}^{N} p_i \, \rho_i \otimes |i\rangle\langle i| \underset{\text{EIGEN.}}{=} \sum_{i=1}^{N} \sum_{j=1}^{d} p_i \lambda_j^{(i)} |v_j^{(i)}\rangle\langle v_j^{(i)}| \otimes |i\rangle\langle i|$$

$$S(\rho_{AB}) = H\left(\{p_i \lambda_j^{(i)}\}\right) = -\sum_{i,j} p_i \lambda_j^{(i)} \log\left(p_i \lambda_j^{(i)}\right) = -\sum_{i,j} p_i \lambda_j^{(i)} \log(p_i) - \sum_{i,j} p_i \lambda_j^{(i)} \log(\lambda_j^{(i)})$$

$$\underset{\left(\sum_j \lambda_j^{(i)} = 1\right)}{=} H(\{p_i\}) + \sum_{i=1}^{N} p_i S(\rho_i)$$

- $$\rho_A = tr_B(\rho_{AB}) = \sum_{i=1}^{N} p_i \rho_i$$

$$S(\rho_A) = S\left(\sum_{i=1}^{N} p_i \rho_i\right)$$

- $$\rho_B = tr_A(\rho_{AB}) = \sum_{i=1}^{N} p_i |i\rangle\langle i|$$

$$S(\rho_B) = H(\{p_i\})$$

- $$S(\rho_{AB}) \underset{\text{SUB-ADD.}}{\leq} S(\rho_A) + S(\rho_B)$$

$$H(\{p_i\}) + \sum_{i=1}^{N} p_i S(\rho_i) \leq S\left(\sum_{i=1}^{N} p_i \rho_i\right) + H(\{p_i\}) \implies \square$$

THEOR.

$$\sum_{i=1}^{N} p_i S(\rho_i) \leq S\left(\sum_{i=1}^{N} p_i \rho_i\right) \leq H(\{p_i\}) + \sum_{i=1}^{N} p_i S(\rho_i)$$

PROOF:

- $S\left(\sum_{i=1}^{N} p_i \rho_i\right) \underset{\rho_i = \sum_{j=1}^{d} \lambda_j |v_j\rangle\langle v_j|}{=} S\left(\sum_{i=1}^{N} \sum_{j=1}^{d} p_i \lambda_j^{(i)} |v_j^{(i)}\rangle\langle v_j^{(i)}|\right)$

- $|\psi\rangle_{AB} := \sum_{i=1}^{N} \sum_{j=1}^{d} \sqrt{p_i \lambda_j^{(i)}}\, |v_j^{(i)}\rangle \otimes \underbrace{|(i,j)\rangle}_{\text{ortonorm. basis}}$

- $S(\rho_A) = S(\mathrm{tr}_B[|\psi\rangle\langle\psi|]) \underset{|\psi\rangle\langle\psi| \text{ pure}}{=} S(\mathrm{tr}_A[|\psi\rangle\langle\psi|]) = S(\rho_B)$

$$S(\rho_A) = S\left(\sum_{i,j} p_i \lambda_j^{(i)} |v_j^{(i)}\rangle\langle v_j^{(i)}|\right) = S\left(\sum_{i=1}^{N} p_i \rho_i\right)$$

$$\Rightarrow S(\rho_B) = S\left(\sum_{i=1}^{N} p_i \rho_i\right)$$

- $\rho_B = \mathrm{tr}_A[|\psi\rangle\langle\psi|] = \mathrm{tr}_A\left(\sum_{i=1}^{N}\sum_{j=1}^{d}\sum_{l=1}^{N}\sum_{k=1}^{d} \sqrt{p_i \lambda_j^{(i)}}\sqrt{p_l \lambda_k^{(l)}}\, |v_j^{(i)}\rangle\langle v_k^{(l)}| \otimes |(i,j)\rangle\langle(l,k)|\right) =$

$$= \sum_{m=1}^{d} \langle m| \otimes \mathbb{1}\left(\sum_{i=1}^{N}\sum_{j=1}^{d}\sum_{l=1}^{N}\sum_{k=1}^{d} \sqrt{p_i \lambda_j^{(i)}}\sqrt{p_l \lambda_k^{(l)}}\, |v_j^{(i)}\rangle\langle v_k^{(l)}| \otimes |(i,j)\rangle\langle(l,k)|\right)|m\rangle \otimes \mathbb{1}$$

$$= \sum_{m=1}^{d} \sum_{i=1}^{N}\sum_{j=1}^{d}\sum_{l=1}^{N}\sum_{k=1}^{d} \sqrt{p_i \lambda_j^{(i)}}\sqrt{p_l \lambda_k^{(l)}}\, \langle m|v_j^{(i)}\rangle\langle v_k^{(l)}|m\rangle\ |(i,j)\rangle\langle(l,k)|$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{d}\sum_{l=1}^{N}\sum_{k=1}^{d} \sqrt{p_i \lambda_j^{(i)}}\sqrt{p_l \lambda_k^{(l)}}\, \langle v_k^{(l)}|v_j^{(i)}\rangle\ |(i,j)\rangle\langle(l,k)|$$

- If I perform ~~a~~ measurement on B (with $\overset{\text{PVM}}{\{\underbrace{|(i,j)\rangle\langle(i,j)|}_{P_{i,j}}\}_{i,j}}$), then the post measurement state will be:

$$\rho_B' = \sum_{i'=1}^{d}\sum_{j'=1}^{N} \mathrm{tr}(\rho_B P_{i',j'})\left(\frac{P_{i',j'}\rho_B P_{i',j'}}{\mathrm{tr}(\rho_B P_{i',j'})}\right)$$

$$= \sum_{i'=1}^{d}\sum_{j'=1}^{N} P_{i',j'}\rho_B P_{i',j'} = \sum_{i',j'} p_{i'} \lambda_{j'}^{(i')} |(i',j')\rangle\langle(i',j')|$$

$$\rho_B' = \sum_{i,j} p_i \lambda_j^{(i)} \, |(i,j)\rangle\langle(i,j)|$$

$$S(\rho_B') = H\left(\{p_i \lambda_j^{(i)}\}\right) = H\left(\{p_i\}\right) + \sum_i p_i \underbrace{\sum_{j=1}^{d} \lambda_j^{(i)} \log\left(\lambda_j^{(i)}\right)}_{\overset{\shortparallel}{S(\rho_i)}}$$

$\boxed{\text{LEMMA}}$ (PROOF LATER) • $\{P_K\}_K$ PVM set $\Rightarrow S\Big(\underbrace{\sum_K P_K \rho P_K}_{\overset{\shortparallel}{\rho'}}\Big) \geq S(\rho)$

$\uparrow$ state after measurement.

• $$S(\rho_B') \geq S(\rho_B) = S\left(\sum_i p_i \rho_i\right)$$

$$\| $$

$$H\left(\{p_i\}\right) + \sum_i p_i S(\rho_i)$$

$\boxed{\text{LEMMA}}$ • $\{P_K\}_K$ PVM set $\Rightarrow S\Big(\underbrace{\sum_K P_K \rho P_K}_{\overset{\shortparallel}{\rho'}}\Big) \geq S(\rho)$

$\uparrow$ state after measurement.

SUB PROOF:

• $$0 \underset{\text{KLEIN INEQ}}{\leq} S(\rho \| \rho') = -S(\rho) - \mathrm{tr}(\rho \log(\rho'))$$

• $$-\mathrm{tr}(\rho \log(\rho')) \geq S(\rho)$$

- $-\mathrm{tr}(\rho \log(\rho')) = -\mathrm{tr}(\rho' \log(\rho')) \equiv S(\rho') \Rightarrow \square$

SUB SUB PROOF:

- $\mathrm{tr}(\rho \log(\rho')) = \mathrm{tr}\left(\sum_i P_i \, \rho \log(\rho')\right) = \sum_i \mathrm{tr}\left(P_i^2 \rho \log(\rho')\right) = \sum_i \mathrm{tr}\left(P_i \rho \log(\rho') P_i\right) =$

  $\left(\text{PVM} \Rightarrow \cdot\, P_i \cdot P_j = \delta_{i,j} P_i \quad \cdot \sum_i P_i = \mathbb{1}\right)$ $\qquad \mathrm{tr}(AB) = \mathrm{tr}(BA)$

- $\log(\rho') P_i = P_i \log(\rho')$

- $\rho' = \sum_j P_j \rho P_j \Rightarrow \rho' P_i = P_i \rho P_i = P_i \rho' \Rightarrow \log(\rho') P_i = P_i \log(\rho')$

  $\log(\rho') = \sum_{k=0}^{\infty} c_k \rho^k$

$= \sum_i \mathrm{tr}\left(P_i \rho P_i \log(\rho')\right) = \sum_i \mathrm{tr}\left(\rho' \log(\rho')\right) \Rightarrow \square$

$\boxed{\text{THEOR. (SSA)}}$ (STRONG SUB-ADDITIVITY)

$$S(\rho_{ABC}) + S(\rho_C) \leq S(\rho_{AC}) + S(\rho_{BC})$$

with $\rho_{AB} = \mathrm{tr}_C(\rho_{ABC})$, $\rho_{BC} = \mathrm{tr}_A(\rho_{ABC})$, $\rho_A = \mathrm{tr}_{BC}(\rho_{ABC})$, - - - -

$\boxed{\text{OBS}}$ STRONG SUB-ADDITIVITY $\Rightarrow$ SUB-ADDITIVITY

PROOF:

- $\rho_{ABC} := \rho_{AB} \otimes |0\rangle\langle 0|$

$$\underbrace{S(\rho_{ABC})}_{\substack{\| \\ S(\rho_{AB}) + \underbrace{S(|0\rangle\langle 0|)}_{\substack{\| \leftarrow \text{PURE} \\ 0}}}} + \underbrace{S(\rho_C)}_{\substack{\| \\ S(|0\rangle\langle 0|) \\ \| \\ 0}} \leq \underbrace{S(\rho_{AC})}_{\substack{\| \\ S(\rho_A \otimes |0\rangle\langle 0|) \\ \| \\ S(\rho_A) + \underbrace{S(|0\rangle\langle 0|)}_{\substack{\| \\ 0}}}} + \underbrace{S(\rho_{BC})}_{\substack{\| \\ S(\rho_B \otimes |0\rangle\langle 0|) \\ \| \\ S(\rho_B) + \underbrace{S(|0\rangle\langle 0|)}_{\substack{\| \\ 0}}}}$$

PRODUCT STATE $\Rightarrow$

$\Rightarrow S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B) \Rightarrow \square$

PROP.

- $\rho^{\otimes n} := \overbrace{\rho \otimes \cdots \otimes \rho}^{n\text{-times}}$

- $S(\rho^{\otimes n}) \underset{\substack{\uparrow \\ \text{SUB-ADD.} \\ \text{(PRODUCT)}}}{=} n S(\rho)$

- $S(\rho^{\otimes n} \| \sigma^{\otimes n}) = n S(\rho \| \sigma)$

PROOF:

$$S(\rho^{\otimes n} \| \sigma^{\otimes n}) = -S(\rho^{\otimes n}) + \mathrm{tr}\left(\rho^{\otimes n} \log(\sigma^{\otimes n})\right) =$$

$$\underset{\substack{\uparrow \\ \left(\begin{array}{l} \cdot\ S(\rho^{\otimes n}) = n S(\rho) \\ \cdot\ \log(A \otimes B) = \log A \otimes \mathbb{1} + \mathbb{1} \otimes \log(B) \end{array} \right)}}{=} -n S(\rho) + n\, \mathrm{tr}(\rho \log(\sigma)) = n S(\rho \| \sigma).$$

3. **... and the von-Neumann entropy**

For any state $\rho \in \mathcal{D}(\mathcal{H})$ with $\dim\mathcal{H} = d$ the von-Neumann entropy is defined as $S(\rho) = -\operatorname{Tr}(\rho \log \rho)$. Throughout this problem, if the global state being referred to is clear, we will denote entropies of the reduced states using the corresponding Hilbert space as an argument, e.g. the entropy of a state ρ_{AB} reduced on subsystem A is denoted $S(A)$.

a) Show that $0 \leq S(\rho)$ with equality if and only if ρ is pure. (One can also show the upper bound $S(\rho) \leq \log d$.)

b) Show that the von-Neumann entropy is *subadditive* in the sense that if two distinct systems A and B have a joint quantum state ρ^{AB} then $S(A,B) \leq S(A) + S(B)$, with equality if $\rho_{AB} = \rho_A \otimes \rho_B$.

Hint: You may use the inequality $S(\rho) \leq -\operatorname{Tr}[\rho \log \sigma]$ for an arbitrary quantum state σ and that for two matrices A and B, $\log(A \otimes B) = \log(A) \otimes \mathbb{1} + \mathbb{1} \otimes \log(B)$.

c) Suppose that $p = (p_i)_i$ is a probability vector and the states ρ_i are mutually orthogonal. Show that

$$S\left(\sum_i p_i \rho_i\right) = H(p) + \sum_i p_i S(\rho_i).$$

and use this result to infer that

$$S\left(\sum_i p_i \rho_i \otimes |i\rangle\langle i|\right) = H(p) + \sum_i p_i S(\rho_i),$$

where $\langle i|j\rangle = \delta_{ij}$ and the ρ_i are arbitrary quantum states.

d) Use the results from b) and c) to infer that the von-Neumann entropy S is concave, that is, $S(\sum_i p_i \rho_i) \geq \sum_i p_i S(\rho_i)$ for a probability distribution $\{p_i\}$.

e) let Ω_{AB} be the maximally entangled state on two Hilbert spaces of equal dimension d, i.e. $\Omega = |\Omega\rangle\langle\Omega|$ with

$$|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle\,. \tag{3}$$

Compute $S(A|B)$. What do you conclude?

A) PROVED BEFORE

B) PROVED BEFORE

c) $\cdot\, S\left(\sum_i p_i \rho_i\right) = S\left(\sum_i p_i \lambda_i^{(j)} \underbrace{|\psi_i^{(j)}\rangle\langle\psi_i^{(j)}|}_{\text{orthog. eigenvectors. because } \rho_i \text{ orthogonals.}}\right) =$

$$= H\left(\{P_i \lambda_i^{(j)}\}\right) = -\sum_{i,j} P_i \lambda_i^{(j)} \log\left(P_i \lambda_i^{(j)}\right) =$$

$$= -\sum_i p_i \underbrace{\left(\sum_j \lambda_i^{(j)}\right)}_{=1} \log(p_i) - \sum_i p_i \sum_j \lambda_i^{(j)} \log\left(\lambda_i^{(j)}\right) =$$

$$= -H\left(\{p_i\}\right) + \sum_i p_i S(\rho_i)$$

- Take $\sum_i p_i \rho_i \otimes \underbrace{|i\rangle\langle i|}_{\text{ort. basis}}$ $\Rightarrow$ $\{\rho_i \otimes |i\rangle\langle i|\}_i$ is ortogon. set.

$$\Rightarrow S\left(\sum_i p_i \rho_i\right) = -H\left(\{p_i\}\right) + \sum_i p_i \underbrace{S\left(\rho_i \otimes |i\rangle\langle i|\right)}_{= S(\rho_i) + \underbrace{S(|i\rangle\langle i|)}_{=0}}$$

D) Seen! $S(A|B) := S(A,B) - S(B)$

$$= \underbrace{S(|\Omega\rangle\langle\Omega|)}_{=0} - S(B) = -\log(d_B) < 0$$

$$S(B) = S(\rho_B) = S\left(\frac{\mathbb{1}}{d_B}\right)$$

Not like in the classical case!