1. **General teleportation schemes**

In the lecture you saw a teleportation scheme using a maximally entangled state shared by Alice and Bob. In this exercise we will generalise this setting to teleportation schemes with higher local dimensions.

We begin by reformulating the qubit teleportation scheme in terms of Bell-basis measurements. The Bell basis for two qubits is given by

$$|\Phi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), |\Phi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),$$
$$|\Phi_2\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), |\Phi_3\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

a) Show that the Bell basis can be prepared starting from $|\Phi_0\rangle$ using local Pauli operations on one subsystem only.

- $|\phi_0\rangle = \mathbb{1} \otimes \mathbb{1} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

- $|\phi_1\rangle = \mathbb{1} \otimes Z \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \mathbb{1} \otimes Z |\phi_0\rangle$

  $Z|1\rangle = -|1\rangle$
  $Z|0\rangle = +|0\rangle$

- $|\phi_2\rangle = \mathbb{1} \otimes X \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

  $X|1\rangle = |0\rangle$
  $X|0\rangle = |1\rangle$

- $|\phi_3\rangle = \mathbb{1} \otimes (XZ) \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = \mathbb{1} \otimes XZ \, |\phi_0\rangle$

In the lecture, you saw the scheme in which Alice holds a pure state $|\psi\rangle$ and shares a maximally entangled state $|\omega\rangle$ with Bob. I.e. the full state is $|\psi\rangle_{A_1} \otimes |\omega\rangle_{A_2B}$. Alice then measures her two qubits in a maximally entangled basis such as the Bell basis and communicates her measurement result to Bob. Bob can now recover the state $|\psi\rangle$, which was originally held by Alice, on his subsystem after a local unitary operation which depends upon the information received from Alice. The state has thus been teleported from Alice to Bob, i.e. Bob now holds a system in the same state as one previously held by Alice even though they did not explicitly send the physical system to each other. Only classical information was exchanged.

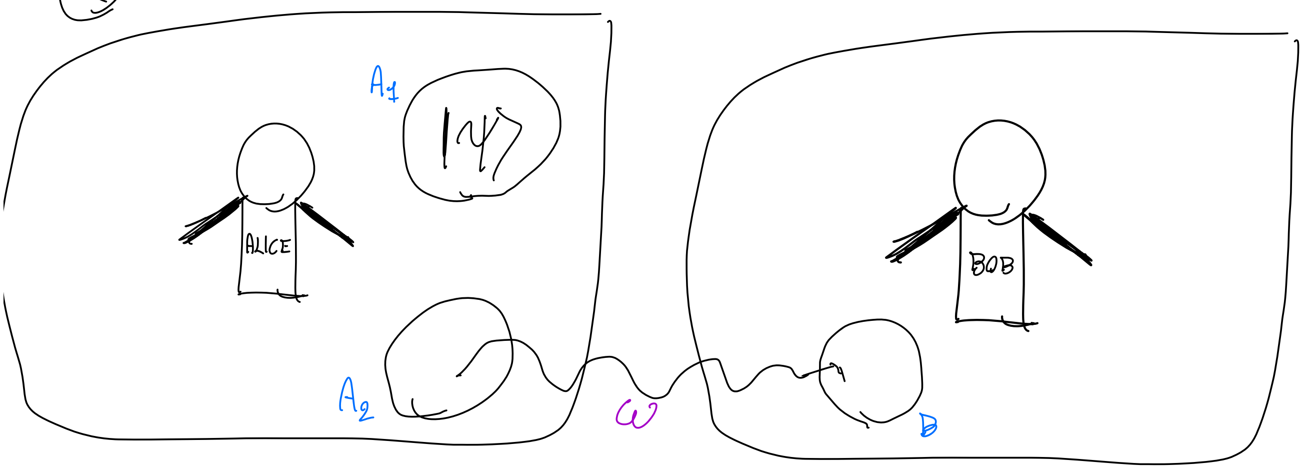
This formulation generalises to a d-dimensional teleportation scheme in which Alice and Bob share a maximally entangled state $|\omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle$. As above the scheme is based on measuring in a maximally entangled orthonormal basis set $\{|\Psi_\alpha\rangle\}_{\alpha=1}^{d^2}$, i.e., an orthonormal basis for which $\mathrm{Tr}_1[|\Psi_\alpha\rangle\langle\Psi_\alpha|] = \frac{\mathbb{1}_d}{d} = \mathrm{Tr}_2[|\Psi_\alpha\rangle\langle\Psi_\alpha|]$.

There exist several constructions of linearly independent sets $\{U^\alpha\}_{\alpha=1}^{d^2}$ of d^2 trace-wise orthogonal unitary operators $U^\alpha \in U(d)$,

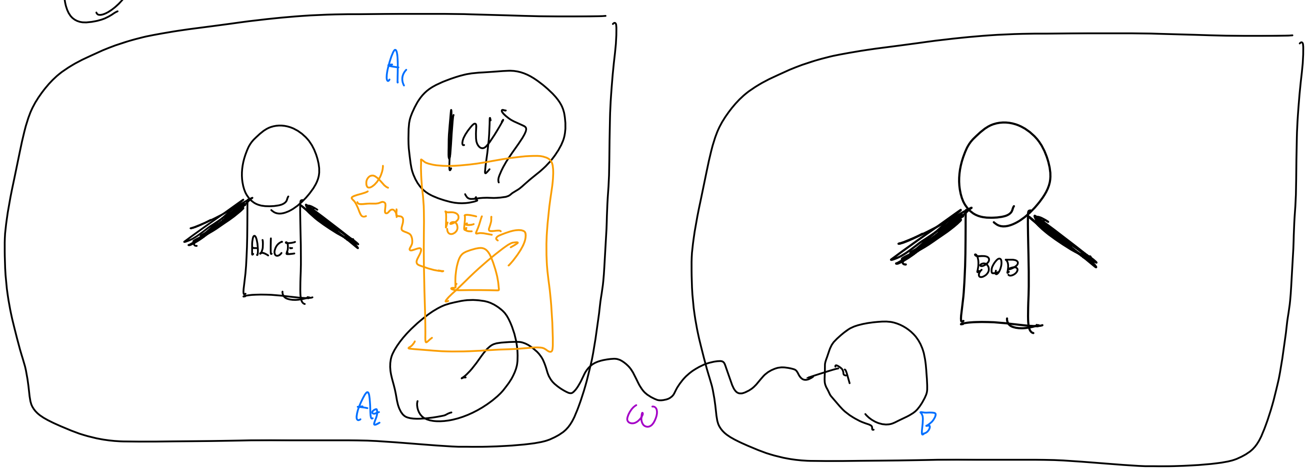
$$\mathrm{Tr}[U^{\alpha\dagger} U^\beta] = \mathrm{Tr}[U^{\beta\dagger} U^\alpha] = \delta_{\alpha\beta} \mathrm{Tr}[\mathbb{1}]$$

for all α and β. In the following, we just assume the existence of such a set.

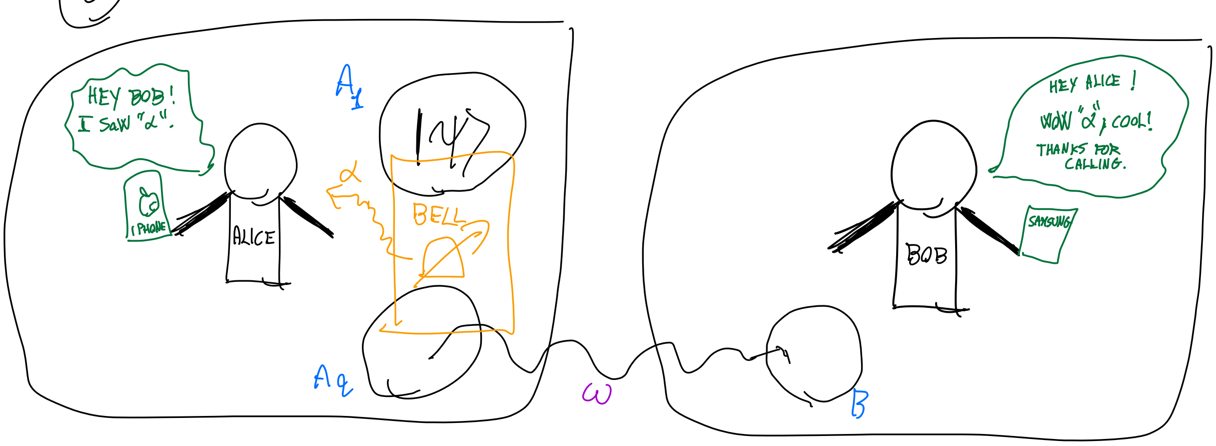
1



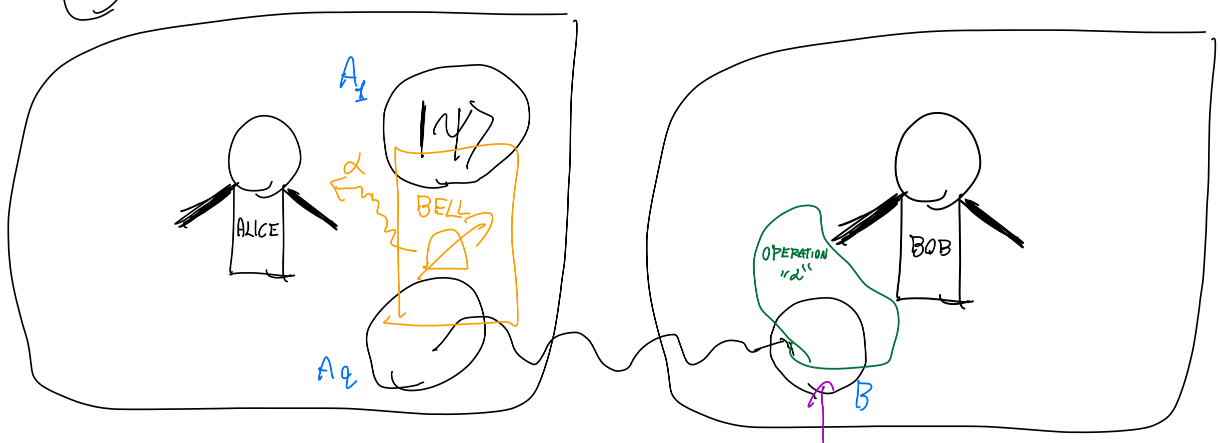
2 Alice performs a "Bell" measurement and sees the outcome "α".



3 ALICE CALLS BOB WITH HER PHONE AND SHE COMMUNICATES THE OUTCOME "α".



(4) BOB PERFORMS AN OPERATION BASED ON "α".



(3) NOW THE STATE OF BOB IS $|\psi\rangle$.

b) Show that such a set $\{U^\alpha\}_{\alpha=1}^{d^2}$ gives rise to a maximally entangled basis set by setting

$$|\Psi_\alpha\rangle = U^\alpha \otimes \mathbb{1}\,|\omega\rangle\,.$$

- $|\Psi_\alpha\rangle$ is maximally entangled i.e. $\rho_B = \frac{\mathbb{1}}{d}$, $\rho_A = \frac{\mathbb{1}}{d}$:

$$\rho_B = tr_A\left(|\Psi_\alpha\rangle\langle\Psi_\alpha|\right) = tr_A\left(\underbrace{U^\alpha \otimes \mathbb{1}\,(|\omega\rangle\langle\omega|)\,U^{+\alpha}}_{U^{+\alpha}U^\alpha = \mathbb{1}_A} \otimes \mathbb{1}\right) =$$

$$= tr_A\left(|\omega\rangle\langle\omega|\right) = \underbrace{\sum_{i,j} tr_A\left(|ii\rangle\langle jj|\right)\frac{1}{d}}_{\frac{1}{d}\sum_i |i\rangle\langle i|} = \frac{\mathbb{1}}{d}$$

$|\omega\rangle = \sum_{i=1}^{d} \frac{|ii\rangle}{\sqrt{d}}$, $tr_A\left(|ii\rangle\langle jj|\right) = \delta_{i,j}$

$$tr_A(U\otimes\mathbb{1}\; O\; U^+\otimes\mathbb{1}) = tr_A(O)$$

$$tr_A(U\otimes\mathbb{1}\; O\; U^+\otimes\mathbb{1}) = \sum_{i,j,k,l} \langle i,j|O|k,l\rangle\, tr_A\left(U\otimes\mathbb{1}\; |i\rangle\langle k| \otimes |j\rangle\langle l|\; U^+\otimes\mathbb{1}\right)$$

$$O = \sum_{i,j,k,l} \langle i,j|O|k,l\rangle\, |i,j\rangle\langle k,l|, \quad |i,j\rangle\langle k,l| = |i\rangle\langle k| \otimes |j\rangle\langle l|$$

$$= \sum_{i,j,k,l} \langle i,j|O|k,l\rangle\, tr\left(U\,|i\rangle\langle k|\,U^+\right) |j\rangle\langle l|, \quad tr\left(U\,|i\rangle\langle k|\,U^+\right) = tr\left(|i\rangle\langle k|\right)$$

$$= tr_A(O)$$

- Similarly $\rho_A = \frac{\mathbb{1}}{d}$.

The maximally entangled state $|w\rangle$ has the following properties, which are important for quantum teleportation scheme.

TRANSPOSE-TRICK.

c) Show that for an arbitrary unitary $U \in U(d)$, $(U \otimes \mathbb{1})|\omega\rangle = (\mathbb{1} \otimes U^T)|\omega\rangle$, where the transpose is taken with respect to the basis of the maximally entangled state.

$$U \otimes \mathbb{1} |\omega\rangle = \frac{1}{\sqrt{d}} U \otimes \mathbb{1} \sum_{i=1}^{d} |ii\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} U|i\rangle \otimes |i\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} \sum_{s=1}^{d} U_{is} |s\rangle \otimes |i\rangle$$

$$= \frac{1}{\sqrt{d}} \sum_{i=1}^{d} \sum_{s=1}^{d} |s\rangle \otimes (U^t)_{si} |i\rangle = \mathbb{1} \otimes U^t \frac{1}{\sqrt{d}} \sum_{s=1}^{d} |s\rangle \otimes |s\rangle = \mathbb{1} \otimes U^t |\omega\rangle$$

d) Show that for an arbitrary pure state of Alice $|\phi\rangle_A$, $(\langle\phi|_A \otimes \mathbb{1}_B)|\omega\rangle = \frac{1}{\sqrt{d}} |\phi^*\rangle_B$ and $\langle\omega|(|\phi\rangle_A \otimes \mathbb{1}_B) = \frac{1}{\sqrt{d}} \langle\phi^*|_B$, where $|\phi^*\rangle$ is the complex conjugate of $|\phi\rangle$ w.r.t. the same basis as the transpose in c).

- $$(\langle\phi| \otimes \mathbb{1}_B)|\omega\rangle = (\langle\phi| \otimes \mathbb{1}) \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle \otimes |i\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} \langle\phi|i\rangle |i\rangle$$

$$= \frac{1}{\sqrt{d}} \sum_{i=1}^{d} (\langle i|\phi\rangle)^* |i\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} \langle i||\phi\rangle^* |i\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle\langle i||\phi\rangle^* = \frac{1}{\sqrt{d}} |\phi\rangle^*$$

$$|i\rangle^* = |i\rangle, \quad |i\rangle := \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{-TH}$$

$$\sum_{i=1}^{d} |i\rangle\langle i| = \mathbb{1}$$

- $$\langle\omega|(|\phi\rangle \otimes \mathbb{1}_B) = \frac{1}{\sqrt{d}} \langle\phi|^*$$

Now consider the setting in which Alice and Bob share the state $|\omega\rangle_{AB}$ and Alice measures her part of the system in the basis $\{|\Psi_\alpha\rangle\}_{AA'}$ to send her state $|\psi\rangle_{A'}$ to Bob.

e) Insert the resolution of the identity $\sum_\alpha |\Psi_\alpha\rangle\langle\Psi_\alpha|_{AA'}$ and use the result from (c) and (d) to show that $|\psi\rangle_{A'}|\omega\rangle_{AB} = \frac{1}{d}\sum_\alpha |\Psi_\alpha\rangle_{AA'} \otimes (U^\alpha)^*_B |\psi\rangle_B$. Then, describe how to perfom d-dimensional quantum teleportation.

$$|\psi\rangle_{A'} |\omega\rangle_{AB} = \sum_\alpha |\Psi_\alpha\rangle_{AA'} \langle\Psi_\alpha|_{AA'} |\psi\rangle_{A'} |\omega\rangle_{AB} =$$

$$= \sum_\alpha |\Psi_\alpha\rangle_{AA'} \langle\omega|_{AA'} U^+_{\alpha\,A} \otimes \mathbb{1}_{A'} |\psi\rangle_{A'} |\omega\rangle_{AB} =$$

$$= \sum_\alpha |\Psi_\alpha\rangle_{AA'} \langle\omega|_{AA'} \left(\mathbb{1}_A \otimes |\psi\rangle_{A'}\right) U^+_{\alpha_A} \otimes \mathbb{1}_{A'} |\omega\rangle_{AB} =$$

$$\underset{\uparrow}{=} \frac{1}{\sqrt{d}} \sum_\alpha |\Psi_\alpha\rangle_{AA'} \langle\psi|^*_A \; U^+_{\alpha_A} \otimes \mathbb{1}_{A'} |\omega_{AB}\rangle$$

$$\langle\omega|_{AA'} \left(\mathbb{1}_A \otimes |\psi\rangle_{A'}\right) \underset{\substack{\uparrow \\ d)}}{=} \langle\psi|^*_A \frac{1}{\sqrt{d}}$$

$$\underset{\substack{\uparrow \\ c)}}{=} \frac{1}{\sqrt{d}} \sum_\alpha |\Psi_\alpha\rangle_{AA'} \langle\psi|^*_A \; \mathbb{1}_A \otimes \mathbb{1}_{A'} \otimes U^*_{\alpha_B} |\omega_{AB}\rangle$$

$$= \frac{1}{\sqrt{d}} \sum_\alpha |\Psi_\alpha\rangle_{AA'} \; \mathbb{1}_A \otimes \mathbb{1}_{A'} \otimes U^*_{\alpha_B} \underbrace{\left(\langle\psi|^*_A \otimes \mathbb{1}_B\right) |\omega_{AB}\rangle}_{\substack{d) \to \, || \\ \frac{1}{\sqrt{d}} |\psi\rangle_B}}$$

$$= \frac{1}{d} \sum_\alpha |\Psi_\alpha\rangle_{AA'} \otimes U^*_{\alpha_B} |\psi\rangle_B$$

- Alice performs the $\{|\Psi_\alpha\rangle\langle\Psi_\alpha|\}_{\alpha=1,\dots,d^2}$ measurement getting an outcome $\tilde{\alpha} \in \{1,\dots,d^2\}$.
- The state before measurement is $\frac{1}{d} \sum_\alpha |\Psi_\alpha\rangle_{AA'} \otimes U^*_{\alpha_B} |\psi\rangle_B$

  $\Rightarrow$ after measurement the state is $|\Psi_{\tilde{\alpha}}\rangle_{AA'} \otimes U^*_{\tilde{\alpha}_B} |\psi\rangle_B$
- Alice calls by phone Bob and communicates the outcome $\tilde{\alpha}$.

- At this point Bob applies $\left(U^+_\alpha\right)^*$ on his subsystem. $\Rightarrow \underbrace{\left(U^+_{\tilde{\alpha}}\right)^* U^*_{\tilde{\alpha}}}_{\substack{|| \\ \mathbb{1}}} |\psi\rangle_B$

2. **From ℓ_p to Schatten norms, to trace distance** (10 points: $1+1+1+1+1+1+1+1+1+1$)
In quantum information we deal with a handful of different matrix spaces such as the set of quantum states.For quantitative statements we have to equip these spaces with distance measures. Depending on the application and context different distance measures have the desired operational meaning.

A prominent role is played by the so called *Schatten p-norms*. But to set the stage we first introduce their analogue on vector spaces, namely *ℓ_p-norms*. For $1 \leq p < \infty$ the ℓ_p-norm on the complex vector space $\mathbb{C}^n$ is defined as

$$||\bullet||_{\ell_p} : x \mapsto ||x||_{\ell_p} := \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}},$$

and the ℓ_∞-norm as

$$||\bullet||_{\ell_\infty} : x \mapsto ||x||_{\ell_\infty} := \lim_{p \to \infty} ||x||_{\ell_p}.$$

a) Show that $||\bullet||_{\ell_\infty} = \max_{1 \leq i \leq n} |x_i|$.

$$\lim_{p\to+\infty} ||X||_{\ell_p} = \lim_{p\to+\infty} \left(\max_j |x_j|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{N} \frac{|x_i|^p}{\max_j |x_j|^p} \right)$$

$$= \max_j |x_j| \underbrace{\lim_{p\to+\infty} \left(1 + \sum_{\substack{i=1 \\ i \neq \operatorname{argmax}(x)}}^{N} \frac{|x_i|^p}{\max_j |x_j|^p} \right)}_{= 1}$$

It is not hard (although tedious) to show that $||\bullet||_{\ell_p}$ satisfies all properties of a norm (it is positive definite, absolutely homogeneous, subadditive aka triangle inequality). Schatten p-norms are defined for linear operators acting on a (finite-dimensional) vector space $\mathcal{V}$ in a similar manner, namely

$$\begin{aligned} \|\bullet\|_p &: L(\mathcal{V}) \to [0, \infty) \\ &O \mapsto \|O\|_p = \left(\mathrm{Tr}\left[|O|^p\right]\right)^{\frac{1}{p}} \end{aligned}$$

where $|O| = \sqrt{O^\dagger O}$.

b) Show that $\|O\|_p = ||\boldsymbol{\sigma}_O||_{\ell_p}$ where $\boldsymbol{\sigma}_O = (\sigma_O(1), \ldots, \sigma_O(n))$ are the singular values of O.

Hint: start by writing the singular value decomposition $O = U\Sigma V$ in Dirac (bra-ket) notation, then write $O^\dagger O$ and apply the definition of p norm.

$$\|O\|_p := \left(\mathrm{tr}\left(|O|^p\right)\right)^{\frac{1}{p}} = \mathrm{tr}\left(\left(\sqrt{O^\dagger O}\right)^p\right)^{1/p} = \mathrm{tr}\left(\left(\sqrt{V D^\dagger U^\dagger U D V^\dagger}\right)\right) =$$

SVD (singular value decomposition): $\exists\, U, V \in U(d)$ such that $O = UDV^\dagger$ with $D \geq 0$ diagonal. $D = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_d \end{pmatrix}$

$$= \mathrm{tr}\left(\left(\sqrt{V D^2 V^\dagger}\right)^p\right)^{\frac{1}{p}} = \mathrm{tr}\left(\left(V\sqrt{D^2}\,V^\dagger\right)^p\right)^{\frac{1}{p}} = \mathrm{tr}\left(\left(V D V^\dagger\right)^p\right)^{1/p} = \mathrm{tr}\left(V D^p V^\dagger\right)^{\frac{1}{p}}$$

- $UU^\dagger = \mathbb{1}$
- $D^\dagger = D$

$\sqrt{VD^2V^\dagger} = V\sqrt{D^2}\,V^\dagger$; $(V\sqrt{D^2}\,V^\dagger)^2 = VD^2V^\dagger$

$D \geq 0$

$$= \mathrm{tr}\left(D^p\right)^{1/p} = \left(\sum_{i=1}^d (D_{ii})^p\right)^{1/p} = \|\sigma_O\|_p$$

- $VV^\dagger = \mathbb{1}$
- $\mathrm{tr}(AB) = \mathrm{tr}(BA)$

$D_{ii} = \sigma_O(i)$

Bonus: If $O = O^\dagger$, $\|O\|_p = \left(\sum_{i=1}^d |\lambda_i|^p\right)^{1/p}$

λ_i: eigenvalues of O.

Alternative proof:

$$O = UDV^\dagger = \sum_{i=1}^d \sigma_i\, |u_i\rangle\langle v_i|$$

$$\sum_{i=1}^d \sum_{s=1}^d (UDV^\dagger)_{is}\, |i\rangle\langle s| = \sum_{i,s=1} \left(\sum_{\ell=1}^d U_{i\ell} D_{\ell\ell} V^\dagger_{\ell s}\right) |i\rangle\langle s| = \sum_{\ell=1}^d \underbrace{\sum_{i=1}^d |i\rangle\langle i|}_{\mathbb{1}} U|\ell\rangle D_{\ell\ell} \langle \ell| V^\dagger \underbrace{\sum_{s=1}^d |s\rangle\langle s|}_{\mathbb{1}} =$$

$U_{i\ell} = \langle i|U|\ell\rangle$
$V^\dagger_{\ell s} = \langle \ell|V^\dagger|s\rangle$

$$= \sum_{\ell=1}^d \underbrace{D_{\ell\ell}}_{\sigma_\ell} \underbrace{U|\ell\rangle}_{|u_\ell\rangle} \underbrace{\langle \ell|V^\dagger}_{\langle v_\ell|} = \sum_{\ell=1}^d \sigma_\ell\, |u_\ell\rangle\langle v_\ell|$$

- $$O^\dagger O = \left(\sum_{i=1}^d \sigma_i^* \,|v_i\rangle\langle u_i|\right)\sum_{s=1}^d \sigma_s\, |u_s\rangle\langle v_s| = \sum_{i=1}^d |\sigma_i|^2\, |v_i\rangle\langle v_i|$$

$|\sigma_i|^2 = \sigma_i^2$; $\sigma_i \geq 0$; $\sigma_i \in \mathbb{R}$

- $$\sqrt{O^\dagger O} = \sqrt{\sum_{i=1}^d \sigma_i^2\, |v_i\rangle\langle v_i|} = \sum_{i=1}^d |\sigma_i|\, |v_i\rangle\langle v_i| \Rightarrow \left(\sqrt{O^\dagger O}\right)^p = \sum_{i=1}^d |\sigma_i|^p\, |v_i\rangle\langle v_i|$$

$|\sigma_i| = \sigma_i \Leftarrow \sigma_i \geq 0$

$$\Rightarrow \mathrm{tr}\left(\left(\sqrt{O^\dagger O}\right)^p\right) = \sum_{i=1}^d |\sigma_i|^p \Rightarrow \underline{\text{II}}$$

c) Show that for any operator A the following holds:

$$||A||_p = ||UAV^\dagger||_p$$

for every unitaries U, V.

$$\bullet \; ||UAV^\dagger||_p = \left(tr\left[\left(\sqrt{(UAU^\dagger)^\dagger(UAU^\dagger)}\right)^p\right]\right)^{1/p} = \left(tr\left[\left(\sqrt{UA^\dagger AU^\dagger}\right)^p\right]\right)^{1/p} =$$

$$= \left(tr\left[\left(U\sqrt{A^\dagger A}\,U^\dagger\right)^p\right]\right)^{1/p} = \left(tr\left[U\left(\sqrt{A^\dagger A}\right)^p U^\dagger\right]\right)^{1/p} =$$

$\uparrow$

$\sqrt{UA^\dagger AU^\dagger} = U\sqrt{A^\dagger A}\,U^\dagger$

$$= \left(tr\left[\left(\sqrt{A^\dagger A}\right)^p\right]\right)^{1/p} = ||A||_p$$

$\uparrow$

- $tr(CD) = tr(DC)$
- $UU^\dagger = \mathbb{1}$

d) Show that for any operator A, B the following holds:

$$||A \otimes B||_p = ||A||_p ||B||_p \tag{7}$$

$$\|A\otimes B\|_p = \|U_A D_A V_A^\dagger \otimes U_B D_B V_B^\dagger\|_p = \|(U_A\otimes U_B)(D_A\otimes D_B)(V_A^\dagger\otimes V_B^\dagger)\|_p =$$

• $A = U_A D_A V_A^\dagger$

SVD

• $B = U_B D_B V_B^\dagger$

$$\overset{c)}{=} \|D_A\otimes D_B\|_p = \left(\mathrm{tr}\left[\left(\sqrt{\underbrace{D_A^\dagger D_A}_{D_A^2} \otimes \underbrace{D_B^\dagger D_B}_{D_B^2}}\right)^p\right]\right)^{1/p} = \mathrm{tr}\left[\left(\sqrt{D_A^\dagger D_A}\otimes\sqrt{D_B^\dagger D_B}\right)^p\right]^{1/p} =$$

• $\sqrt{A}\otimes\sqrt{B} = \sqrt{A\otimes B}$

$(\sqrt{A}\otimes\sqrt{B})^2 = (\sqrt{A})^2\otimes(\sqrt{B})^2 = A\otimes B$

$$= \mathrm{tr}\left[\left(\sqrt{D_A^\dagger D_A}\right)^p\otimes\left(\sqrt{D_B^\dagger D_B}\right)^p\right]^{1/p} = \mathrm{tr}\left[\left(\sqrt{D_A^\dagger D_A}\right)^p\right]\mathrm{tr}\left[\left(\sqrt{D_B^\dagger D_B}\right)^p\right]^{1/p} = \|D_A\|_p\|D_B\|_p = \|A\|_p\|B\|_p$$

$\mathrm{tr}(A\otimes B) = \mathrm{tr}(A)\,\mathrm{tr}(B)$

• $A = V_A D_A V_A^\dagger$

• c)

A notable special case is $p = 1$, that is $\|O\|_1 = \mathrm{Tr}\,[|O|]$, which turns out to give a useful measure of distinguishability between quantum states, called *trace distance* $\|\rho - \sigma\|_1$. The remaining exercises of this sheet will focus on this.

e) Show that $0 \le \|\rho - \sigma\|_1 \le 2$ for any pair of density matrices.

• $\|\rho - \sigma\|_1 \geq 0$

$\|\cdot\|_1 \geq 0$

• $\|\rho - \sigma\|_1 \leq \|\rho\|_1 + \|\sigma\|_1 = 2$

TRIANGLE IN.

$\|\rho\|_1 = \mathrm{tr}(\sqrt{\rho^2}) = \mathrm{tr}(\rho) = 1$

$\rho \geq 0$

In the following, we will prove that the normalized trace distance provides an achievable upper bound for the probability of obtaining the same outcome if *any* measurement (POVM) is performed on ρ vs σ. Suppose Alice flips a coin and, depending on the result, sends either ρ or σ to Bob. Bob wants to perform a measurement that will tell him which one of the two states he has. To this end, he implements a POVM with two operators, M_0 and M_1, such that the outcome 0 means the state is ρ and the outcome 1 means the state is σ.

f) Show that the probability that Bob successfully determines which state he has is

$$P_{\text{success}} = \frac{1}{2}(1 + \mathrm{Tr}[M_0(\rho - \sigma)]) \tag{8}$$

$$P_{succ} = P(\rho)\cdot P(\text{guessing } \rho \text{ given } \rho) + P(\sigma)\cdot P(\text{guessing given } \sigma)$$

$$= \frac{1}{2}\cdot \underbrace{P(\text{guessing } \rho \text{ given } \rho)}_{= tr(M_0 \rho)} + \frac{1}{2}\cdot \underbrace{P(\text{guessing } \sigma \text{ given } \sigma)}_{= tr(M_1 \sigma)}$$

$$= \frac{1}{2}\left(tr(M_0 \rho) + tr(M_1 \sigma)\right) =$$

$$\underset{\uparrow}{=} \frac{1}{2}\left(tr(M_0\rho) + tr((\mathbb{1} - M_0)\sigma)\right) \underset{\uparrow}{=} \frac{1}{2}\left(1 + tr(M_0(\rho - \sigma))\right)$$

$\{M_0, M_1\}$ POVM $\Rightarrow$ $\cdot\ M_0 + M_1 = \mathbb{1}$
$\cdot\ M_0, M_1 \geq 0$

$tr(\sigma) = 1$

The trace distance measures the optimal probability of distinguishing the states, in equations this reads

$$\frac{1}{2}\|\rho - \sigma\|_1 = \max_{0 \leq M \leq \mathbb{1}} \mathrm{Tr}\left[M(\rho - \sigma)\right]. \tag{10}$$

We will prove this in a few steps.

g) Write $|\rho - \sigma|$ in terms of the the positive and negative parts of $\rho - \sigma$, P and Q.

$$\rho - \sigma \underset{\substack{\uparrow \\ \text{eigendecomp.}}}{=} \sum_{i=1}^{d} \lambda_i |v_i\rangle\langle v_i| = \underbrace{\sum_{\lambda_i \geq 0} \lambda_i |v_i\rangle\langle v_i|}_{= P} - \underbrace{\sum_{\lambda_i \leq 0} |\lambda_i| |v_i\rangle\langle v_i|}_{= Q} = P - Q$$

with $P \geq 0$, $Q \geq 0$

$$\cdot\ |\rho - \sigma| = \sqrt{(\rho-\sigma)^\dagger(\rho - \sigma)} = \sqrt{\sum_i |\lambda_i|^2 |v_i\rangle\langle v_i|} = \sum_i |\lambda_i| |v_i\rangle\langle v_i| = P + Q$$

h) Show that $\mathrm{Tr}[P] = \mathrm{Tr}[Q]$.

$$\rho - \sigma = P - Q \Rightarrow \underbrace{\mathrm{tr}(\rho - \sigma)}_{=0} = \mathrm{tr}(P - Q)$$

$\mathrm{tr}(\sigma) = 1$
$\mathrm{tr}(\rho) = 1$

$$\Rightarrow \mathrm{tr}(P) = \mathrm{tr}(Q)$$

i) Consider the projector on the support of P, Π_P. Use the previous three points to show that $\mathrm{Tr}[\Pi_P(\rho - \sigma)] = \frac{1}{2}\|\rho - \sigma\|_1$.
Hint: try to write each side in terms of $\mathrm{Tr}[P]$.

$$\bullet\ \mathrm{tr}[\Pi_P(\rho - \sigma)] = \mathrm{tr}[\Pi_P(P - Q)] = \mathrm{tr}(\Pi_P P) = \mathrm{tr}(P)$$

$\bullet\ \Pi_P Q = 0$ $\qquad$ $\bullet\ \Pi_P P = P$

$$\left(\begin{array}{l} \Pi_P = \sum_{\lambda_i > 0} |v_i\rangle\langle v_i| \\ Q = \sum_{\lambda_i < 0} \lambda_i |v_i\rangle\langle v_i| \end{array} \right)$$

$$\bullet\ \frac{1}{2}\|\rho - \sigma\|_1 = \frac{1}{2}\mathrm{tr}(|\rho - \sigma|) = \frac{1}{2}(\mathrm{tr}(P) + \mathrm{tr}(Q)) = \mathrm{tr}(P)$$

$|\rho - \sigma| = P + Q$ $\qquad$ $\mathrm{tr}(P) = \mathrm{tr}(Q)$

$\Rightarrow \square$

We are almost done: Π_P achieves our upper bound. We just need to show that Π_P is also the optimal POVM element:

j) Show that any positive operator M such that $M \leq \mathbb{I}$ will obey

$$\mathrm{Tr}\left[M\left(\rho-\sigma\right)\right] \leq \frac{1}{2}\left\|\rho-\sigma\right\|_1 = \mathrm{Tr}\left[\Pi_P\left(\rho-\sigma\right)\right]. \tag{16}$$

Hint: use again $\rho - \sigma = P - Q$ and inequalities for the trace of positive operators we've seen in a previous sheet.

- $$\mathrm{tr}\left(M\left(\rho-\sigma\right)\right) = \mathrm{Tr}\left(M\left(P-Q\right)\right) = \mathrm{tr}(MP) - \mathrm{tr}(MQ) \underset{\uparrow}{\leq} \mathrm{tr}(MP) =$$

$\cdot\ \mathrm{tr}(MQ) \geq 0$ $\uparrow$ $\cdot\ Q \geq 0$ $\cdot\ M \geq 0$

$$\underset{\uparrow}{\leq} \mathrm{tr}(P) \underset{\uparrow\, i)}{=} \frac{1}{2}\|\rho-\sigma\|_1$$

$\cdot\ \mathrm{tr}(MP) \leq \mathrm{tr}(P)$ $\Leftarrow$ $\left(\cdot\ M \leq \mathbb{1} \Leftrightarrow (\mathbb{1}-M) \geq 0 \quad \cdot\ P \geq 0 \right)$

$$\mathrm{tr}\left(M\left(\rho-\sigma\right)\right) \leq \frac{1}{2}\|\rho-\sigma\|_1$$

$$P_{succ} \underset{\uparrow\, f)}{=} \frac{1}{2}\left(1 + \mathrm{tr}\left(M_0\left(\rho-\sigma\right)\right)\right) \leq \frac{1}{2}\left(1 + \frac{1}{2}\|\rho-\sigma\|_1\right) = \underbrace{\frac{1}{2}\left(1 + \mathrm{tr}\left(\Pi_P\left(\rho-\sigma\right)\right)\right)}_{P_{succ}^{OPT}}$$

Finally, we can give an operational interpretation to orthogonal states: the following points are to show that Bob can perfectly distinguish ρ and σ if and only if they are orthogonal, i.e. $\mathrm{Tr}(\rho\sigma) = 0$.

k) Show that if $\mathrm{Tr}(\rho\sigma) = 0$, then $||\rho - \sigma||_1 = 2$. *Hint: first, show that* $\mathrm{Tr}(\rho\sigma) = 0 \implies \rho\sigma = 0$

- $$\mathrm{Tr}(\rho\sigma) \underset{\uparrow\, \rho = \sum_i \lambda_i |V_i\rangle\langle V_i|}{=} \sum_i \lambda_{i_\rho} \langle V_{i_\rho}|\sigma|V_{i_\rho}\rangle \underset{\uparrow\, \sigma = \sum_j \lambda_{j_\sigma} |V_{j_\sigma}\rangle\langle V_{j_\sigma}|}{=} \sum_{i,j} \lambda_{i_\rho}\lambda_{j_\sigma} |\langle V_{i_\rho}|V_{j_\sigma}\rangle|^2$$

$$= \sum_{\substack{i,j \\ \lambda_{i_\rho}\neq 0 \\ \lambda_{j_\sigma}\neq 0}} \lambda_{i_\rho}\lambda_{j_\sigma} |\langle V_{i_\rho}|V_{j_\sigma}\rangle|^2 = 0 \underset{\uparrow\, \rho \geq 0,\ \sigma \geq 0}{\Rightarrow} |\langle V_{i_\rho}|V_{j_\sigma}\rangle|^2 = 0 \Rightarrow \langle V_{i_\rho}|V_{j_\sigma}\rangle = 0 \quad \forall i_\rho, j_\sigma$$

$\Rightarrow (\rho\sigma = 0)$

$$\Rightarrow \rho - \sigma = \sum_i \lambda_{i\rho} |V_{i\rho}\rangle\langle V_{i\rho}| - \sum_i \lambda_{i\sigma} |V_{i\sigma}\rangle\langle V_{i\sigma}|$$

$$\cdot |\rho - \sigma| = \sum_i |\lambda_{i\rho}| |V_{i\rho}\rangle\langle V_{i\rho}| + \sum_i |\lambda_{i\sigma}| |V_{i\sigma}\rangle\langle V_{i\sigma}|$$

$\cdot |V_{i\rho}\rangle$ are eigenstates of $\rho-\sigma$ with $\lambda_{i\rho}$ eigenvalue

$\cdot |V_{i\sigma}\rangle$ '' ---- '' $\rho-\sigma$ with $\lambda_{i\sigma}$ eigenvalue

$$= \underbrace{\sum_i \lambda_{i\rho} |V_{i\rho}\rangle\langle V_{i\rho}|}_{\rho} + \underbrace{\sum_i \lambda_{i\sigma} |V_{i\sigma}\rangle\langle V_{i\sigma}|}_{\sigma} = \rho + \sigma$$

$\rho \geq 0 \Rightarrow \lambda_{i\rho} \geq 0$

$\sigma \geq 0 \Rightarrow \lambda_{i\sigma} \geq 0$

- So, if $tr(\rho\sigma) = 0 \Rightarrow \|\rho - \sigma\|_1 = tr(|\rho - \sigma|) = tr(\rho + \sigma) = 2$

- $P_{succ}^{OPT} = \frac{1}{2}\left(1 + \frac{1}{2}\|\rho - \sigma\|_1\right) \underset{\|\rho-\sigma\|_1 \leq 2}{\leq} \frac{1}{2}(1 + 1) = 1$

It's saturated when $\|\rho - \sigma\|_1 = 0$

We have shown that if $tr(\rho\sigma) = 0$ $(\Rightarrow \|\rho - \sigma\|_1 = 0) \Rightarrow P_{succ}^{OPT} = 1$

l) Conversely, show that $||\rho - \sigma||_1 = 2$ implies $\text{Tr}(\rho\sigma) = 0$. What does this imply for the probability of distinguishing ρ and σ?

Hint: recall that $||\rho - \sigma||_1 = 2\,\text{Tr}(\Pi_P(\rho - \sigma))$, *use this to show that* $\text{Tr}(\Pi_P\rho) = 1$ *and* $\text{Tr}(\Pi_P\sigma) = 0$.

$\cdot\, 2 = \|\rho - \sigma\|_1 = 2\,Tr(\Pi_P(\rho - \sigma)) \Rightarrow \boxed{tr(\Pi_P \rho) = 1 + tr(\Pi_P \sigma)} = (\star_A)$

- Remember that if $(\rho - \sigma) \underset{EIGEN.}{=} \sum_i \lambda_i |V_i\rangle\langle V_i| \Rightarrow \Pi_P = \sum_{i=1}^{\#\{\lambda_i \geq 0\}} |V_i\rangle\langle V_i| \Rightarrow \Pi_P \underset{(\mathbb{1} - \Pi_P) \geq 0}{\leq} \mathbb{1}$

But $tr(\Pi_P \rho) \leq 1$ and $tr(\Pi_P \sigma) \geq 0 \overset{\star_A}{\Rightarrow}$

$$tr(\Pi_P \rho) \overset{\uparrow}{=} tr((\mathbb{1} - \mathbb{1} + \Pi_P)\rho) = \underset{1}{tr(\rho)} - tr((\mathbb{1} - \Pi_P)\rho) \leq 1$$

$tr((\mathbb{1} - \Pi_P)\rho) \underset{\rho \geq 0}{\geq} 0$

(This is saturated when $tr((\mathbb{1} - \Pi_P)\rho) = 0$)

$\Rightarrow$ $\boxed{tr(\Pi_p \rho) = 1}$, $\boxed{tr(\Pi_p \sigma) = 0}$

$\Rightarrow \cdot\ tr\left((\mathbb{1} - \Pi_p)\, \rho \right) = 0 \quad , \quad tr(\sigma \Pi_p) = 0$

$\Rightarrow \Pi_p \rho = \rho, \quad \sigma \Pi_p = 0 \quad \Rightarrow tr(\rho\sigma) = tr(\Pi_p \rho \sigma) = tr(\rho\, \underbrace{\sigma \Pi_p}_{0}) = 0$

$\Pi_p \rho = \rho$

$\cdot\ \mathbb{1} - \Pi_p \geq 0,\ \rho \geq 0 \Rightarrow (\mathbb{1} - \Pi_p)\rho = 0$

$\cdot\ \Pi_p \geq 0,\ \sigma \geq 0 \Rightarrow \Pi_p \sigma = 0$

m) Using the previous points argue that the optimal probability of distinguishing two states remains unchanged if we consider only projective measurements instead of general POVM.

We proved that:

$P_{succ}^{OPT} = \frac{1}{2}\left(1 + tr(\Pi_p(\rho - \sigma))\right)$ where $\Pi_p = \sum_{\substack{i \\ \lambda_i \geq 0}} |V_i\rangle\langle V_i|$

eigenvalue ≥ 0 of $\rho - \sigma$

eigenvectors of $\rho - \sigma$.

$\Rightarrow$ Note that $\{\Pi_p, \underbrace{\mathbb{1} - \Pi_p}_{\Pi_Q}\}$ is a PVM since $\Pi_p \Pi_Q = 0$ and $\Pi_p^2 = \Pi_p$.
$\Pi_Q^2 = \Pi_Q$

So the max of P_{succ} over all the POVMs is achieved by a PVM.

$\Rightarrow \max_{POVM} (P_{succ}) = \max_{PVM} (P_{succ})$,