1. **Tensor products**
The configuration space of a quantum system with multiple degrees of freedom is described by the tensor product of the Hilbert spaces of each degree of freedom. In the following exercise we will familiarise ourselves with the construction of tensor product spaces.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces with basis $B_1 = \{|i\rangle_1\}_{i=1}^{d}$ and $B_2 = \{|j\rangle_2\}_{i=1}^{D}$, respectively. One can construct a new vector space $\mathcal{H}_1 \otimes \mathcal{H}_2$ by using the set of tuples $B_1 \times B_2 = \{(|i\rangle_1, |j\rangle_2) : |i\rangle_1 \in B_1, |j\rangle_2 \in B_2\}$ as a basis. The basis elements $(|i\rangle_1, |j\rangle_2)$ are also typically denoted by $|i\rangle|j\rangle$, $|i,j\rangle$ or $|i\rangle\otimes|j\rangle$. The last notation can be extended to a bilinear composition $\otimes : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2$ by defining

$$|\psi\rangle \otimes |\phi\rangle := \sum_{i=1}^{d}\sum_{j=1}^{D} \langle i|\psi\rangle \langle j|\phi\rangle |i,j\rangle . \tag{1}$$

This automatically defines a scalar on the tensor product space product by

$$\langle ij|kl\rangle = \delta_{ik}\delta_{jl} \tag{2}$$

Moreover:

COMPLETENESS RELATION:

If $B_1 = \{|i\rangle\}_{i=1}^{d}$ is an ortonormal basis for $\mathcal{H}_1$, then $\sum_{i=1}^{d} |i\rangle\langle i| = \mathbb{1}_{\mathcal{H}_1}$

- If $B_1 = \{|i\rangle\}_{i=1}^{d}$ is an ortonormal basis for $\mathcal{H}_1$, and $B_2 = \{|j\rangle\}_{j=1}^{D}$ is an ortonormal basis for $\mathcal{H}_2$, then

$$\mathbb{1}_{\mathcal{H}_1\otimes\mathcal{H}_2} = \sum_{i=1}^{d}\sum_{j=1}^{D} |i,j\rangle\langle i,j|$$

$$\mathbb{1}_{\mathcal{H}_1\otimes\mathcal{H}_2} = \mathbb{1}_{\mathcal{H}_1} \otimes \mathbb{1}_{\mathcal{H}_2} = \left(\sum_{i=1}^{d} |i\rangle\langle i|\right) \otimes \left(\sum_{j=1}^{D} |j\rangle\langle j|\right) =$$

$$= \sum_{i=1}^{d}\sum_{j=1}^{D} \underbrace{|i\rangle\langle i| \otimes |j\rangle\langle j|}_{\substack{= (|i\rangle\otimes|j\rangle)(\langle i|\otimes\langle j|) \\ = |i,j\rangle\langle i,j|}}$$

$$= \sum_{i=1}^{d}\sum_{j=1}^{D} |i,j\rangle\langle i,j|$$

- Eq (1) is a "good definition" because:

$$|\psi\rangle \otimes |\phi\rangle = (\mathbb{1} \otimes \mathbb{1}) |\psi\rangle \otimes |\phi\rangle =$$

$$= \sum_{i=1}^{d} \sum_{j=1}^{D} \left(|i,j\rangle\langle i,j| \right) |\psi\rangle \otimes |\phi\rangle =$$

$$= \sum_{i=1}^{d} \sum_{j=1}^{D} |i,j\rangle \langle i|\psi\rangle \langle j|\phi\rangle$$

- Now let's start with the first question:

a) What is the dimension of the vector space $\mathcal{H}_1 \otimes \mathcal{H}_2$? What is the Hilbert space of a system of n spin-1/2 particles? What is its dimension?

- $\dim(\mathcal{H}_1 \otimes \mathcal{H}_2)$ = Number of elements of a $\mathcal{H}_1 \otimes \mathcal{H}_2$ basis =

$$\underset{\substack{\uparrow \\ B_{\mathcal{H}_1 \otimes \mathcal{H}_2} = \{|i,j\rangle\}_{\substack{i=1,\dots,d \\ j=1,\dots,D}}}}{=} d \cdot D$$

- The Hilbert space of a spin-$\frac{1}{2}$ particle is $\mathbb{C}^2$ equipped with the hermitian scalar product. In particular, $\dim(\mathbb{C}^2) = 2$ and we denote by $\{|0\rangle, |1\rangle\}$ an orthonormal basis for $\mathbb{C}^2$.

The Hilbert space of n spin-$\frac{1}{2}$ particles is $\underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2}_{n \text{ times}} := (\mathbb{C}^2)^{\otimes n}$

We have $\dim((\mathbb{C}^2)^{\otimes n})$ = Number of elements of a $(\mathbb{C}^2)^{\otimes n}$ basis =

$\underset{\uparrow}{=}$ Number of elements of $\left(B_{(\mathbb{C}^2)^{\otimes n}} \right) = \overbrace{2 \cdots 2}^{n \text{ times}} = 2^n$

We consider the basis $B_{(\mathbb{C}^2)^{\otimes n}} = \{|i_1\rangle \otimes |i_2\rangle \otimes \dots \otimes |i_n\rangle$ with $i_1, \dots, i_n = 0, 1\}$

b) Show that the operation $\otimes : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2$ defined above is bilinear.

We should show that $(\alpha |\psi_1\rangle + \beta |\psi_2\rangle) \otimes |\phi\rangle = \alpha |\psi_1\rangle \otimes |\phi\rangle + \beta |\psi_2\rangle \otimes |\phi\rangle$ using the definition $|\psi\rangle \otimes |\phi\rangle := \sum_{i=1}^{d} \sum_{j=1}^{D} \langle i|\psi\rangle \langle j|\phi\rangle |i, j\rangle$.

We have :

$$(\alpha |\psi_1\rangle + \beta |\psi_2\rangle) \otimes |\phi\rangle = \sum_{i=1}^{d} \sum_{j=1}^{D} \langle i| (\alpha |\psi_1\rangle + \beta |\psi_2\rangle) \langle j|\phi\rangle \, |i, j\rangle =$$

$$= \alpha \sum_{i=1}^{d} \sum_{j=1}^{D} \langle i|\psi_1\rangle \langle j|\phi\rangle \, |i, j\rangle + \beta \langle i|\psi_2\rangle \langle j|\phi\rangle \, |i, j\rangle =$$

$$= \alpha |\psi_1\rangle \otimes |\phi\rangle + \beta |\psi_2\rangle \otimes |\phi\rangle$$

c) Is $\otimes : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2$ surjective? (Please argue.)

$\forall |\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \; \exists \; |\phi_1\rangle \otimes |\phi_2\rangle$ with $|\phi_1\rangle \in \mathcal{H}_1$ and $|\phi_2\rangle \in \mathcal{H}_2$ s.t. $|\psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle$?

No, a COUNTEREXAMPLE is an entangled state e.g. :

$$\frac{|0,0\rangle + |1,1\rangle}{\sqrt{2}}$$

d) Define the tensor product on the level of operators $A, B, C : \mathcal{H}_1 \to \mathcal{H}_1$ acting on vectors $|\phi\rangle, |\psi\rangle \in \mathcal{H}_1$: as

$$(A \otimes B)(|\phi\rangle \otimes |\psi\rangle) = (A|\phi\rangle) \otimes (B|\psi\rangle), \tag{3}$$

show that

(i) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

$$(A \otimes B)(C \otimes D)\, |\phi\rangle \otimes |\psi\rangle \underset{\text{eq.}(3)}{=} A \otimes B \left(C|\phi\rangle \otimes D|\psi\rangle \right) \underset{\text{eq.}(3)}{=} (AC|\phi\rangle) \otimes (BD|\psi\rangle)$$

$$\Rightarrow (A \otimes B)(C \otimes D) = AC \otimes BD$$

2. **Pauli matrices and the Bloch sphere**

The Pauli matrices are one of the most ubiquitous objects in quantum mechanics. They act on the simplest non-trivial Hilbert space $\mathcal{H} = \mathbb{C}^2$.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In this exercise we want to recap their properties.

a) Show that these matrices mutually anticommute, i.e. $AB = -BA$ and that all of them square to the identity.

First of all let's familiarize with the different notations:

We define $|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 1\,|1\rangle\langle 0| + 1\,|0\rangle\langle 1|$; $Y = i\,|1\rangle\langle 0| - i\,|0\rangle\langle 1|$, $Z = 1|0\rangle\langle 0| - 1\,|1\rangle\langle 1|$

- $X^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad Y^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

  $Z^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

- $XY = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = iZ$

  $YX = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -iZ$ $\quad \Rightarrow \quad XY = -YX$

Similarly for $XZ = -iY$ and $YZ = iX$
$ZX = iY \Rightarrow XZ = -ZX$ $\qquad ZY = -iX \Rightarrow YZ = -ZY$

So denoting $P_1 := X$, $P_2 := Y$, $P_3 = Z$;

if $i, j \in \{1, 2, 3\} \Rightarrow P_i P_j = \delta_{ij} + i\,\epsilon_{ijk} P_k$

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In particular if $i \neq j \Rightarrow P_i P_j = i\,\epsilon_{ijk} P_k$ with $k \neq i$ and $k \neq j$

e.g. $XY = iZ$
$YZ = iZ$
$XZ = -iY$

b) Explicitly compute the 4×4 matrices $X \otimes X$, $Z \otimes Z$, $X \otimes Y$ and $Y \otimes X$ in the tensor product basis.

• We'll present 3 ways to do that.

First of all, let us stress something about $A \otimes B$.

• If A is a matrix acting on $\mathcal{H}_1$,
" B " " " " on $\mathcal{H}_2$, $\Rightarrow$ then $A \otimes B$ acts on $\mathcal{H}_1 \otimes \mathcal{H}_2$.

1) We have:

If $\{|i\rangle\}_{i=1}^{d}$ is an orthonorm. basis for $\mathcal{H}_1$ and $\{|j\rangle\}_{j=1}^{D}$ for $\mathcal{H}_2$, then:

$$\boxed{A \otimes B = \sum_{\substack{i=1,\dots,d \\ j=1,\dots,D}} \sum_{\substack{k=1,\dots,d \\ l=1,\dots,D}} \langle i|A|k\rangle\langle j|B|l\rangle \; |i,j\rangle\langle k,l|} \equiv (eq. \bigstar)$$

In fact:

$$A \otimes B = \mathbb{1}_{\mathcal{H}_1 \otimes \mathcal{H}_2} \, A \otimes B \, \mathbb{1}_{\mathcal{H}_1 \otimes \mathcal{H}_2}$$

$$= \sum_{\substack{i=1,\dots,d \\ j=1,\dots,D}} |i,j\rangle\langle i,j| \left(A \otimes B\right) \sum_{\substack{k=1,\dots,d \\ l=1,\dots,D}} |k,l\rangle\langle k,l| =$$

COMPLETENESS RELATION

$$= \sum_{\substack{i=1,\dots,d \\ j=1,\dots,D}} \sum_{\substack{k=1,\dots,d \\ l=1,\dots,D}} \langle i,j| \, A \otimes B \, |k,l\rangle \; |i,j\rangle\langle k,l| =$$

$$= \sum_{\substack{i=1,\dots,d \\ j=1,\dots,D}} \sum_{\substack{k=1,\dots,d \\ l=1,\dots,D}} \langle i|A|k\rangle\langle j|B|l\rangle \; |i,j\rangle\langle k,l|$$

So using eq. $\bigstar$ one can verify explicitly that using the basis $\{|0\rangle, |1\rangle\}$:

$$X \otimes X = 1\,|0,0\rangle\langle 1,1| + 1\,|0,1\rangle\langle 1,0| + 1\,|1,0\rangle\langle 0,1| + 1\,|1,1\rangle\langle 0,0|$$

$$= \begin{array}{c c} & \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array}$$

And similarly for the others $(X\otimes Y, Z\otimes Z, \dots)$.

2) We have $A\otimes B = \begin{pmatrix} A_{11}B & \cdots & A_{1d}B \\ \vdots & & \vdots \\ A_{d1}B & \cdots & A_{dd}B \end{pmatrix}$

where $A_{ij} = \langle i|A|j\rangle$,

and $B = \begin{pmatrix} B_{11} & \cdots & B_{1D} \\ \vdots & & \vdots \\ B_{D1} & \cdots & B_{DD} \end{pmatrix}$ with $B_{ij} = \langle i|B|j\rangle$

So we have:

$$X\otimes Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes Y = \begin{pmatrix} 0\cdot Y & 1\cdot Y \\ 1\cdot Y & 0\cdot Y \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & 1\cdot\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ 1\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

Note that for writing the matrix we used the ordering of the basis $\{ |1,1\rangle, |1,2\rangle, |1,3\rangle, \dots, |1,D\rangle,$
$|2,1\rangle, \dots, |2,D\rangle,$
$\vdots$
$|d,1\rangle, \dots, |d,D\rangle \}$

(This simple Rule can be derived by manipulating eq (*):

$$\boxed{A\otimes B = \sum_{\substack{i=1,\dots,d \\ j=1,\dots,D}} \sum_{\substack{k=1,\dots,d \\ l=1,\dots,D}} \langle i|A|k\rangle\langle j|B|l\rangle \; |i,j\rangle\langle k,l|})$$

Similarly for the others.

3) One can derive the matrix form of $A \otimes B$ just computing $(A \otimes B)|i\rangle \otimes |j\rangle$ $\forall$ element of the basis $|i\rangle \otimes |j\rangle$.

For example $(X \otimes Y)\, |0\rangle \otimes |0\rangle = |1\rangle \otimes (i)|1\rangle = i\,|1\rangle \otimes |1\rangle$

$X \otimes Y \; |0\rangle \otimes |1\rangle = -i\,|1\rangle \otimes |0\rangle$

$X \otimes Y \; |1\rangle \otimes |0\rangle = i\,|0\rangle \otimes |1\rangle$

$X \otimes Y \; |1\rangle \otimes |1\rangle = -i\,|0\rangle \otimes |0\rangle$

So defining $|0\rangle \otimes |0\rangle =: \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $|0,1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $|1,0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $|1,1\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

we have:

$$X \otimes Y = \begin{array}{cc} \begin{array}{cccc} 0,0 & 0,1 & 1,0 & 1,1 \end{array} & \\ \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} & \begin{array}{c} 0,0 \\ 0,1 \\ 1,0 \\ 1,1 \end{array} \end{array}$$

And similarly for the others.

c) Can you express the product XZ again as a Pauli matrix?

Yes, up to "phase" $(-i)$ in fact $XZ = -iY$.

- So let's recap and stress useful properties of Paulis:

PAULI PROPERTIES:

1) $P^2 = \mathbb{1}$

2) $P^\dagger = P$

3) $\mathrm{tr}(P) = 0$

4) $P_1 \neq P_2 \Rightarrow P_1 P_2 = -P_2 P_1$ (e.g. $XY = -YX$)

5) $P_1 \neq P_2 \Rightarrow P_1 P_2 = (\pm i) P_3$ with $P_3 \neq P_1$ and $P_3 \neq P_2$

6) $P_1 \neq P_2 \Longrightarrow \mathrm{tr}(P_1 P_2) = 0$

We now want to use Pauli matrices to study the space of all qubit observables: the hermitian matrices $h(\mathbb{C}^2)$. This space is canonically equipped with the *Hilbert-Schmidt inner product*

$$\langle A, B\rangle := \mathrm{Tr}(AB^\dagger).$$

The norm that is defined by this product is also called *Frobenius norm*. Both will constant companions in this course.

d) Show that with respect to this inner product that the Pauli matrices together with the identity form an orthogonal basis for $h(\mathbb{C}^2)$.

We have A hermitian $\Leftrightarrow A^\dagger = A$

$$\Rightarrow A := \begin{pmatrix} a+ib & c+id \\ e+if & g+ih \end{pmatrix}$$

$$\parallel$$

$$A^\dagger \underset{A^\dagger = (A^T)^*}{=} \begin{pmatrix} a-ib & e-if \\ c-id & g-ih \end{pmatrix}$$

with $a, b, c, d, e, f \in \mathbb{R}$

$$\Rightarrow \begin{cases} b = 0 \\ h = 0 \\ f = -d \\ e = c \end{cases} \Rightarrow A = \begin{pmatrix} a & c+id \\ c-id & g \end{pmatrix}$$

Now :

$$A = \left(\frac{a+g}{2}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \left(\frac{a-g}{2}\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - d\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \underbrace{\left(\frac{a+g}{2}\right)}_{= c_0} \mathbb{1} + \underbrace{\left(\frac{a-g}{2}\right)}_{= c_1} Z + \underbrace{c}_{= c_2} X \underbrace{- d}_{= c_3} Y$$

$$= c_0 \mathbb{1} + c_1 Z + c_2 X + c_3 Y$$

$\Rightarrow$ $X, Y, Z, \mathbb{1}$ are generators of $A = A^{+}$.

- To prove that $X, Y, Z, \mathbb{1}$ is a basis we still need to show that they are linearly independent.

If $A = 0 \Rightarrow A = \begin{pmatrix} a & c+id \\ c-id & g \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow a = c = d = g = 0$

$$\Rightarrow c_0 = c_1 = c_2 = c_3 = 0 \Rightarrow \square$$

- Moreover $X, Y, Z, \mathbb{1}$ are orthogonal, in fact:

If $P_1, P_2 \in \{X, Y, Z\}$ and $P_1 \neq P_2$, we have:

$$\langle P_1, P_2\rangle = tr(P_1^+ P_2) \underset{\substack{\uparrow \\ P_1 = P_1^+}}{=} tr(P_1 P_2) \underset{\substack{\uparrow \\ \cdot\, P_1 \neq P_2 \Rightarrow P_1P_2 = \pm i P_3 \\ \cdot\, tr(P) = 0}}{=} 0$$

If $P_1 \in \{X, Y, Z\}$ and $P_2 = \mathbb{1}$, we have:

$$\langle P_1, \mathbb{1}\rangle = tr(P_1^+) = tr(P_1) = 0$$

e) Find the normalized version of this basis with respect to the Frobenius norm.

Let $P_1 \in \{\mathbb{1}, X, Y, Z\} \Rightarrow$

$$\langle P_1, P_1\rangle = tr(P_1^+ P_1) \underset{\substack{\uparrow \\ P_1^+ = P_1}}{=} tr(P_1 P_1) = tr(P_1^2) \underset{\substack{\uparrow \\ P_1^2 = \mathbb{1}}}{=} tr(\mathbb{1}) = 2$$

$\Rightarrow \quad \frac{X}{\sqrt{2}}, \frac{Y}{\sqrt{2}}, \frac{Z}{\sqrt{2}}, \frac{\mathbb{1}}{\sqrt{2}}$ is an orthonormal basis.

Recall that a density matrix ρ is a hermitian operator with all positive eigenvalues such that $\mathrm{Tr}(\rho) = 1$. We restrict ourselves – for now – to the qubit case.

f) You have shown that the Paulis with the identity form a basis of the hermitian matrices. Prove that, in this basis, the set of density matrices is described by a unit ball $B_1(0) = \{(a, b, c) \in \mathbb{R}^3;\ a^2 + b^2 + c^2 \leq 1\}$ (called *Bloch sphere*).

Recall that ρ is a density matrix $\Longleftrightarrow \begin{cases} \rho \geq 0 \\ \rho = \rho^+ \\ tr(\rho) = 1 \end{cases}$

- $\rho = \rho^{+} \Rightarrow \rho = c_{\mathbb{1}} \frac{\mathbb{1}}{\sqrt{2}} + c_X \frac{X}{\sqrt{2}} + c_Y \frac{Y}{\sqrt{2}} + c_Z \frac{Z}{\sqrt{2}}$

- $tr(\rho) = 1 \Rightarrow \underbrace{tr(\rho)}_{1} = c_{\mathbb{1}} \frac{\overset{2}{tr(\mathbb{1})}}{\sqrt{2}} + c_X \frac{\overset{0}{tr(X)}}{\sqrt{2}} + c_Y \frac{\overset{0}{tr(Y)}}{\sqrt{2}} + c_Z \frac{\overset{0}{tr(Z)}}{\sqrt{2}}$

$$\Rightarrow c_{\mathbb{1}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \rho = \frac{1}{2} \mathbb{1} + c_X \frac{X}{\sqrt{2}} + c_Y \frac{Y}{\sqrt{2}} + c_Z \frac{Z}{\sqrt{2}}$$

- $\rho \geq 0 \Rightarrow tr(\rho^2) \leq 1$

In fact: $\rho^{+} = \rho \Rightarrow \rho = \lambda_1 |v_1\rangle\langle v_1| + \lambda_2 |v_2\rangle\langle v_2| = \sum_i \lambda_i |v_i\rangle\langle v_i|$

ρ DIAGONALIZABLE with eigenvalues λ_1, λ_2 and eigenvectors $|v_1\rangle$ and $|v_2\rangle$

or in general

$\rho \geq 0 \Rightarrow \lambda_i \geq 0$

$\sum_i \lambda_i = \sum_i \lambda_i \, tr(|v_i\rangle\langle v_i|) = tr(\underbrace{\sum_i \lambda_i |v_i\rangle\langle v_i|}_{\rho}) = tr(\rho) = 1 \Rightarrow \sum_i \lambda_i = 1 \underset{\lambda_i \geq 0}{\Longrightarrow} \lambda_i \leq 1$

$\Rightarrow 0 \leq \lambda_i \leq 1 \Rightarrow \lambda_i^2 \leq \lambda_i$

- $1 = tr(\rho) = \sum_i \lambda_i \underset{\lambda_i^2 \leq \lambda_i}{\geq} \sum_i \lambda_i^2 = tr(\rho^2)$

$\rho^2 = \sum_i \lambda_i^2 |v_i\rangle\langle v_i|$

$\Rightarrow tr(\rho^2) = \sum_i \lambda_i^2 \underbrace{tr(|v_i\rangle\langle v_i|)}_{1} = \sum_i \lambda_i^2$

$$\Rightarrow tr(\rho^2) \leq 1$$

- So $tr(\rho^2) \leq 1 \Rightarrow$

$tr(\rho^2) = tr\left(\left(\frac{1}{2} \mathbb{1} + c_X \frac{X}{\sqrt{2}} + c_Y \frac{Y}{\sqrt{2}} + c_Z \frac{Z}{\sqrt{2}}\right)^2\right) =$

$$= tr\left(\left(\frac{1}{2}\mathbb{1} + c_X\frac{X}{\sqrt{2}} + c_Y\frac{Y}{\sqrt{2}} + c_Z\frac{Z}{\sqrt{2}}\right)\left(\frac{1}{2}\mathbb{1} + c_X\frac{X}{\sqrt{2}} + c_Y\frac{Y}{\sqrt{2}} + c_Z\frac{Z}{\sqrt{2}}\right)\right) =$$

$$= \left(\frac{1}{2}\right)^2 tr(\mathbb{1}) + \left(\frac{c_X}{\sqrt{2}}\right)^2 tr(\underbrace{X^2}_{=\mathbb{1}}) + \left(\frac{c_Y}{\sqrt{2}}\right)^2 tr(\underbrace{Y^2}_{=\mathbb{1}}) + \left(\frac{c_Z}{\sqrt{2}}\right)^2 tr(\underbrace{Z^2}_{=\mathbb{1}}) =$$

$$\begin{cases} tr(XY) = i\,tr(Z) = 0 \\ tr(XZ) = 0 \\ \vdots \\ tr(YZ) = 0, \quad tr(X) = tr(Y) = tr(Z) = 0 \end{cases}$$

$$= \frac{1}{2} + c_X^2 + c_Y^2 + c_Z^2 \overset{\text{★2}}{=} \le 1$$

$tr(\mathbb{1}) = 2$ $\qquad$ $tr(\rho^2) \le 1$

$$\Rightarrow c_X^2 + c_Y^2 + c_Z^2 \le \frac{1}{2}$$

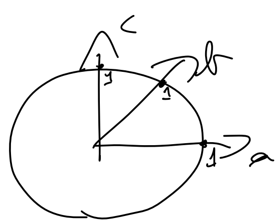
$$\Rightarrow (\sqrt{2}c_X)^2 + (\sqrt{2}c_Y)^2 + (\sqrt{2}c_Z)^2 \le 1$$

So we define $a := \sqrt{2}c_X$

$b := \sqrt{2}c_Y$

$c := \sqrt{2}c_Z$

$$\Rightarrow a^2 + b^2 + c^2 \le 1 \Rightarrow$$



g) Where do the pure states ($\rho = |\psi\rangle\langle\psi|$) live in this ball? Which point corresponds to the maximally mixed state ($\rho = \mathbb{1}/2$)?

$$\rho = |\psi\rangle\langle\psi| \Rightarrow tr(\rho^2) = tr(|\psi\rangle\underbrace{\langle\psi|\psi\rangle}_{=1}\langle\psi|) = tr(|\psi\rangle\langle\psi|) = 1$$

★2: $tr(\rho^2) = \frac{1}{2} + \frac{a^2}{2} + \frac{b^2}{2} + \frac{c^2}{2}$

$\Rightarrow \quad a^2 + b^2 + c^2 = 1 \Rightarrow$ pure states are on the surface.

• $\rho = \frac{1}{2} \Rightarrow c_x = c_y = c_z = 0 \Rightarrow a = b = c = 0$

$\Rightarrow$ center of the ball.

3. **Beam splitters and interferometers**

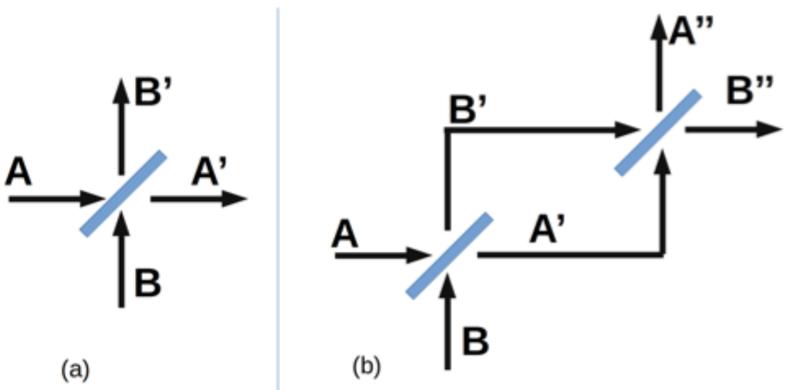


Figure 1: Depiction of a beam splitter

In this exercise we consider a simple example of operations on a qubit consisting of a single photon. To this end we introduce an optical element called *beam splitter*, that has two inputs and two outputs (see Fig. 1(a)). Each input consists of an *optical mode* that can be populated by *photons* (don't worry if these words don't mean much to you at the moment, they should become intuitively clear in the following). For example the state

$$|\bar{0}\rangle = |1\rangle_A |0\rangle_B$$

represents a photon in mode A and no photons in mode B. Similarly

$$|\bar{1}\rangle = |0\rangle_A |1\rangle_B$$

means that the photon is in mode B. In general, we will denote by $|n\rangle_J$ the state describing n photons in mode J. We note $\langle n|n'\rangle = \delta_{n,n'}$. Quantum-mechanically, a beam splitter is given as a *linear, unitary operator* whose restriction to $\mathcal{H} = \text{span}\left(|\bar{0}\rangle, |\bar{1}\rangle\right)$ is represented by the matrix

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

in the same basis.

a) Confirm that $|\bar{0}\rangle$ and $|\bar{1}\rangle$ are orthogonal. The span of these vectors is hence $\mathcal{H} = \mathbb{C}^2$ and can be then treated as an effective qubit.

$\cdot \quad \langle \tilde{0} | \tilde{1} \rangle = \left(\langle 1|_A \langle 0|_B \right) \underbrace{\left(|0\rangle_A \, |1\rangle_B \right)}_{\substack{\shortparallel \\ (|0\rangle_A \otimes |1\rangle_B)}} = \underbrace{\langle 1|0\rangle_A}_{\substack{\shortparallel \\ 0}} \, \underbrace{\langle 0|1\rangle_B}_{\substack{\shortparallel \\ 0}} = 0$

We introduce the number operator

$$N_J = \sum_{n=0}^{\infty} n \, |n\rangle_J \langle n|$$

that "counts" how many photons are in mode J, in the sense that $N_J |n\rangle_J = n |n\rangle_J$. The total number of photons in modes A and B is counted by the operator

$$N_{AB} = N_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes N_B.$$

b) Show that any state $|\psi\rangle = \alpha |\tilde{0}\rangle + \beta |\tilde{1}\rangle$ satisfies $N_{AB} |\psi\rangle = |\psi\rangle$. We then say that the *total photon number* is one.

$$N_{AB} = N_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes N_B =$$

$$= \sum_{n=1}^{\infty} n \left(|n\rangle\langle n| \otimes \mathbb{1} \right) + \sum_{n=1}^{+\infty} n \left(\mathbb{1} \otimes |n\rangle\langle n| \right)$$

$$N_{AB} |\psi\rangle = \alpha \, N_{AB} |\tilde{0}\rangle + \beta \, N_{AB} |\tilde{1}\rangle =$$

$$= \alpha \, N_{AB} |1\rangle \otimes |0\rangle + \beta \, N_{AB} |0\rangle \otimes |1\rangle =$$

$$= \alpha \sum_{n=1}^{\infty} n \left(|n\rangle\langle n| \otimes \mathbb{1} \right) |1\rangle \otimes |0\rangle + \alpha \sum_{n=1}^{+\infty} n \left(\mathbb{1} \otimes |n\rangle\langle n| \right) |1\rangle \otimes |0\rangle$$

$$+ \beta \sum_{n=1}^{\infty} n \left(|n\rangle\langle n| \otimes \mathbb{1} \right) |0\rangle \otimes |1\rangle + \beta \sum_{n=1}^{+\infty} n \left(\mathbb{1} \otimes |n\rangle\langle n| \right) |0\rangle \otimes |1\rangle =$$

$$= \alpha \underbrace{|1\rangle \otimes |0\rangle}_{|\tilde{0}\rangle} + \alpha \cdot 0 + \beta \cdot 0 + \beta \underbrace{|0\rangle \otimes |1\rangle}_{\substack{\shortparallel \\ |\tilde{1}\rangle}} = \alpha |\tilde{0}\rangle + \beta |\tilde{1}\rangle = |\psi\rangle$$

$\sum_{n=1}^{+\infty} n \langle n|0\rangle = 0$

$\sum_{n=1}^{+\infty} n \langle n|1\rangle = 1$

c) What can we say about the total photon number of the state $S |\tilde{0}\rangle$?

$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \mathbb{1} + \frac{i}{\sqrt{2}} X \quad \Rightarrow S|\tilde{0}\rangle = \frac{1}{\sqrt{2}} |\tilde{0}\rangle + \frac{i}{\sqrt{2}} |\tilde{1}\rangle$

(where $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = |\tilde{0}\rangle$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = |\tilde{1}\rangle$)

$\underset{\substack{\uparrow \\ b)}}{\Rightarrow} N_{AB} \, S|\tilde{0}\rangle = S|\tilde{0}\rangle$

d) Show that the operators $|0\rangle_A \langle 0| \otimes \mathbb{I}_B, |1\rangle_A \langle 1| \otimes \mathbb{I}_B$ are projectors. What can we say about the probability of having a photon in mode A if we have the state $|\tilde{0}\rangle$?

- $\left(|0\rangle\langle 0|_A \otimes \mathbb{1}_B\right)^2 = \left(|0\rangle \underbrace{\langle 0|0\rangle}_{=1} \langle 0|\right) \otimes \mathbb{1} = |0\rangle\langle 0| \otimes \mathbb{1}_B$

- $\left(|1\rangle\langle 1|_A \otimes \mathbb{1}_B\right)^2 = \left(|1\rangle \underbrace{\langle 1|1\rangle}_{=1} \langle 1|\right) \otimes \mathbb{1} = |1\rangle\langle 1| \otimes \mathbb{1}_B$

- Prob($n_A = 1$ if we have $|\tilde{0}\rangle$) = Prob($n_A=1$ if we have $|\tilde{0}\rangle = \underbrace{|1\rangle}_{n_A} \otimes |0\rangle$) = 1

e) Upon input $|\tilde{0}\rangle$, what is the probability of having one photon in mode A' at the output of the beam splitter? And in mode B'? What are these probabilities if we instead input $|\tilde{1}\rangle$?

$$|\psi\rangle = S|\tilde{0}\rangle = \left(\frac{1}{\sqrt{2}}\mathbb{1} + \frac{i}{\sqrt{2}}X\right)|\tilde{0}\rangle = \frac{1}{\sqrt{2}}|\tilde{0}\rangle + \frac{i}{\sqrt{2}}|\tilde{1}\rangle$$

$$\text{Prob}(n_A = 1) = |\langle\tilde{0}|\psi\rangle|^2 = \frac{1}{2}$$

$$\text{Prob}(n_B = 1) = |\langle\tilde{1}|\psi\rangle|^2 = \frac{1}{2}$$

$$|\phi\rangle = S|\tilde{0}\rangle = \left(\frac{1}{\sqrt{2}}\mathbb{1} + \frac{i}{\sqrt{2}}X\right)|\tilde{1}\rangle = \frac{1}{\sqrt{2}}|\tilde{1}\rangle + \frac{i}{\sqrt{2}}|\tilde{0}\rangle$$

$$\text{Prob}(n_A = 1) = |\langle\tilde{0}|\phi\rangle|^2 = \frac{1}{2}$$

$$\text{Prob}(n_B = 1) = |\langle\tilde{1}|\phi\rangle|^2 = \frac{1}{2}$$

We can interpret the previous result saying that a single photon, coming in at any input of the beam splitter, has probability 1/2 to be transmitted and probability 1/2 to be reflected by S. Beam splitters can be combined to obtain more general *interferometers*. For example, we can imagine to send the outputs of S, A' and B' into the inputs of another beam splitter (see Fig.1(b)).

f) Calculate the probability to detect one photon at the output A'' of the second beam splitter.

We have the state $S^2|\tilde{0}\rangle$.

$$S^2|\hat{0}\rangle = S\left(\frac{1}{\sqrt{2}}|\hat{0}\rangle + \frac{i}{\sqrt{2}}|\hat{1}\rangle\right) =$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|\hat{0}\rangle + \frac{i}{\sqrt{2}}|\hat{1}\rangle\right) + \frac{i}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|\hat{1}\rangle + \frac{i}{\sqrt{2}}|\tilde{0}\rangle\right)$$

$$= i|\hat{1}\rangle$$

We could also have observed that:

$$S^2 = \left(\frac{1}{\sqrt{2}}\mathbb{1} + \frac{i}{\sqrt{2}}X\right)\left(\frac{1}{\sqrt{2}}\mathbb{1} + \frac{i}{\sqrt{2}}X\right) =$$

$$= \frac{\mathbb{1}}{2} + 2\cdot i\frac{X}{2} - \underbrace{\frac{X^2}{2}}_{\frac{\mathbb{1}}{2}} = iX$$

So $S^2|\tilde{0}\rangle = iX|\tilde{0}\rangle = i|\hat{1}\rangle \rightsquigarrow \mathrm{Prob}(n_A=1) = 0$

$S^2|\hat{1}\rangle = i|\tilde{0}\rangle \rightsquigarrow \mathrm{Prob}(n_A=1) = 1$

Now assume that we repeat this construction, namely we keep adding beam splitters that each take the output state of the preceding one as their inputs.

g) Calculate the detection probabilities of the output states of such an interferometer with N beam splitters, where $N \in \mathbb{N}$.

$$S = \frac{\mathbb{1}}{\sqrt{2}} + i\frac{X}{\sqrt{2}}$$

$$S^2 = i X$$

$$S^3 = S S^2 = i\left(\frac{X}{\sqrt{2}} + i\frac{\mathbb{1}}{\sqrt{2}}\right)$$

$$S^4 = S^2 S^2 = -X^2 = -\mathbb{1}$$

$$S^5 = S S^4 = -S$$

$$S^6 = S^2 S^4 = -S^2$$

$$S^7 = S^3 S^4 = -S^3$$

$$S^8 = S^4 S^4 = \mathbb{1}$$

$$\Rightarrow \quad S^N = \begin{cases} \pm\mathbb{1} & N \equiv 0 \mod 4 \\ \pm S & N \equiv 1 \mod 4 \\ \pm S^2 = \pm i X & N \equiv 2 \mod 4 \\ \pm S^3 & N \equiv 3 \mod 4 \end{cases}$$

$$S^N|\hat{0}\rangle = \begin{cases} \pm\mathbb{1}|\hat{0}\rangle & N \equiv 0 \mod 4 \\ \pm S|\hat{0}\rangle & N \equiv 1 \mod 4 \\ \pm i X|\hat{0}\rangle & N \equiv 2 \mod 4 \\ \pm S^3|\hat{0}\rangle & N \equiv 3 \mod 4 \end{cases}$$

$$= \begin{cases} \pm|\hat{0}\rangle & N \equiv 0 \mod 4 \\ \pm\frac{1}{\sqrt{2}}\left(|\hat{0}\rangle + i|\hat{1}\rangle\right) & N \equiv 1 \mod 4 \\ \pm i|\hat{1}\rangle & N \equiv 2 \mod 4 \\ \pm\frac{i}{\sqrt{2}}\left(|\hat{1}\rangle + i|\hat{0}\rangle\right) & N \equiv 3 \mod 4 \end{cases}$$

$$\Rightarrow \quad \mathrm{Prob}\left(h_A = 1 \text{ given } S^N|\hat{0}\rangle\right) = \begin{cases} 1 & N \equiv 0 \mod 4 \\ \frac{1}{2} & N \equiv 1 \mod 4 \\ 0 & N \equiv 2 \mod 4 \\ \frac{1}{2} & N \equiv 3 \mod 4 \end{cases}$$

$$|\langle\hat{0}|S^N|\hat{0}\rangle|^2$$

h) Imagine that someone performs a measurement on mode B' but does not tell us whether they detected a photon or not. How can we describe the state of the two modes?

The state before measurement is $|\psi\rangle = \alpha|\tilde{0}\rangle + \beta|\tilde{1}\rangle$
(density matrix)

If we perform a measurement but we don't know the outcome, we can describe the system with a mixed state:

$$\rho_{\substack{\text{AFTER}\\\text{MEAS}}} = \underbrace{\text{Prob}(|\tilde{0}\rangle \text{ given } |\psi\rangle)}_{|\langle\tilde{0}|\psi\rangle|^2} |\tilde{0}\rangle\langle\tilde{0}| + \underbrace{\text{Prob}(|\tilde{1}\rangle \text{ given } |\psi\rangle)}_{|\langle\tilde{1}|\psi\rangle|^2} |\tilde{1}\rangle\langle\tilde{1}| =$$

$$= |\alpha|^2 |\tilde{0}\rangle\langle\tilde{0}| + |\beta|^2 |\tilde{1}\rangle\langle\tilde{1}| .$$

Freie Universität Berlin
Tutorials on Quantum Information Theory
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Problem Sheet 0
Warm-up

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1. **Tensor products**
The configuration space of a quantum system with multiple degrees of freedom is described by the tensor product of the Hilbert spaces of each degree of freedom. In the following exercise we will familiarise ourselves with the construction of tensor product spaces.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces with basis $B_1 = \{|i\rangle_1\}_{i=1}^d$ and $B_2 = \{|j\rangle_2\}_{i=1}^D$, respectively. One can construct a new vector space $\mathcal{H}_1 \otimes \mathcal{H}_2$ by using the set of tuples $B_1 \times B_2 = \{(|i\rangle_1, |j\rangle_2) : |i\rangle_1 \in B_1, |j\rangle_2 \in B_2\}$ as a basis. The basis elements $(|i\rangle_1, |j\rangle_2)$ are also typically denoted by $|i\rangle |j\rangle$, $|i, j\rangle$ or $|i\rangle \otimes |j\rangle$. The last notation can be extended to a bilinear composition $\otimes : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2$ by defining

$$|\psi\rangle \otimes |\phi\rangle \coloneqq \sum_{i=1}^{d} \sum_{j=1}^{D} \langle i|\psi\rangle \langle j|\phi\rangle |i, j\rangle . \tag{1}$$

This automatically defines a scalar on the tensor product space product by

$$\langle ij|kl\rangle = \delta_{ik}\delta_{jl} \tag{2}$$

a) What is the dimension of the vector space $\mathcal{H}_1 \otimes \mathcal{H}_2$? What is the Hilbert space of a system of n spin-1/2 particles? What is its dimension?
b) Show that the operation $\otimes : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2$ defined above is bilinear.
c) Is $\otimes : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2$ surjective? (Please argue.)
d) Define the tensor product on the level of operators $A, B, C : \mathcal{H}_1 \to \mathcal{H}_1$ acting on vectors $|\phi\rangle, |\psi\rangle \in \mathcal{H}_1$: as

$$(A \otimes B)(|\phi\rangle \otimes |\psi\rangle) = (A|\phi\rangle) \otimes (B|\psi\rangle), \tag{3}$$

show that

(i) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

2. **Pauli matrices and the Bloch sphere**
The Pauli matrices are one of the most ubiquitous objects in quantum mechanics. They act on the simplest non-trivial Hilbert space $\mathcal{H} = \mathbb{C}^2$.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In this exercise we want to recap their properties.

a) Show that these matrices mutually anticommute, i.e. $AB = -BA$ and that all of them square to the identity.
b) Explicitly compute the 4×4 matrices $X \otimes X$, $Z \otimes Z$, $X \otimes Y$ and $Y \otimes X$ in the tensor product basis.
c) Can you express the product XZ again as a Pauli matrix?

We now want to use Pauli matrices to study the space of all qubit observables: the hermitian matrices $h(\mathbb{C}^2)$. This space is canonically equipped with the *Hilbert-Schmidt inner product*

$$\langle A, B\rangle := \mathrm{Tr}(AB^\dagger).$$

The norm that is defined by this product is also called *Frobenius norm*. Both will constant companions in this course.

d) Show that with respect to this inner product that the Pauli matrices together with the identity form an orthogonal basis for $h(\mathbb{C}^2)$.

e) Find the normalized version of this basis with respect to the Frobenius norm.

Recall that a density matrix ρ is a hermitian operator with all positive eigenvalues such that $\mathrm{Tr}(\rho) = 1$. We restrict ourselves – for now – to the qubit case.

f) You have shown that the Paulis with the identity form a basis of the hermitian matrices. Prove that, in this basis, the set of density matrices is described by a unit ball $B_1(0) = \{(a, b, c) \in \mathbb{R}^3;\ a^2 + b^2 + c^2 \leq 1\}$ (called *Bloch sphere*).

g) Where do the pure states ($\rho = |\psi\rangle\langle\psi|$) live in this ball? Which point corresponds to the maximally mixed state ($\rho = \mathbb{1}/2$)?

3. **Beam splitters and interferometers**

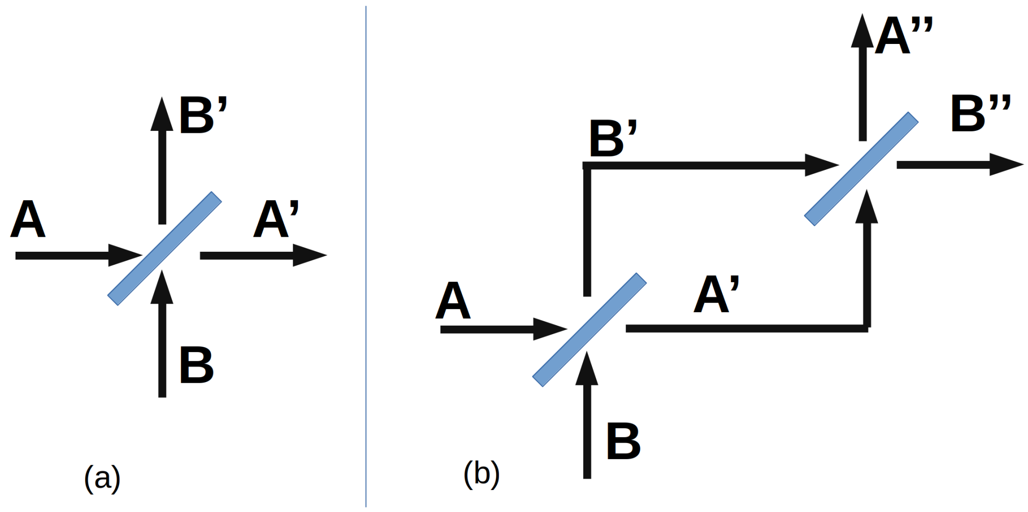

Figure 1: Depiction of a beam splitter

In this exercise we consider a simple example of operations on a qubit consisting of a single photon. To this end we introduce an optical element called *beam splitter*, that has two inputs and two outputs (see Fig. 1(a)). Each input consists of an *optical mode* that can be populated by *photons* (don't worry if these words don't mean much to you at the moment, they should become intuitively clear in the following). For example the state

$$|\tilde{0}\rangle = |1\rangle_A |0\rangle_B$$

represents a photon in mode A and no photons in mode B. Similarly

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means that the photon is in mode B. In general, we will denote by $|n\rangle_J$ the state describing n photons in mode J. We note $\langle n|n'\rangle = \delta_{n,n'}$. Quantum-mechanically, a beam splitter is given as a *linear, unitary operator* whose restriction to $\mathcal{H} = \text{span}\left(\left|\tilde{0}\right\rangle, \left|\tilde{1}\right\rangle\right)$ is represented by the matrix

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

in the same basis.

a) Confirm that $\left|\tilde{0}\right\rangle$ and $\left|\tilde{1}\right\rangle$ are orthogonal. The span of these vectors is hence $\mathcal{H} = \mathbb{C}^2$ and can be then treated as an effective qubit.

We introduce the number operator

$$N_J = \sum_{n=0}^{\infty} n \left|n\right\rangle_J \left\langle n\right|$$

that "counts" how many photons are in mode J, in the sense that $N_J \left|n\right\rangle_J = n \left|n\right\rangle_J$. The total number of photons in modes A and B is counted by the operator

$$N_{AB} = N_A \otimes \mathbb{I}_B + \mathbb{I}_A \otimes N_B.$$

b) Show that any state $\left|\psi\right\rangle = \alpha\left|\tilde{0}\right\rangle + \beta\left|\tilde{1}\right\rangle$ satisfies $N_{AB}\left|\psi\right\rangle = \left|\psi\right\rangle$. We then say that the *total photon number* is one.

c) What can we say about the total photon number of the state $S\left|\tilde{0}\right\rangle$?

d) Show that the operators $\left|0\right\rangle_A \left\langle 0\right| \otimes \mathbb{I}_B, \left|1\right\rangle_A \left\langle 1\right| \otimes \mathbb{I}_B$ are projectors. What can we say about the probability of having a photon in mode A if we have the state $\left|\tilde{0}\right\rangle$?

e) Upon input $\left|\tilde{0}\right\rangle$, what is the probability of having one photon in mode A' at the output of the beam splitter? And in mode B'? What are these probabilities if we instead input $\left|\tilde{1}\right\rangle$?

We can interpret the previous result saying that a single photon, coming in at any input of the beam splitter, has probability $1/2$ to be transmitted and probability $1/2$ to be reflected by S. Beam splitters can be combined to obtain more general *interferometers*. For example, we can imagine to send the outputs of S, A' and B' into the inputs of another beam splitter (see Fig.1(b)).

f) Calculate the probability to detect one photon at the output A'' of the second beam splitter.

Now assume that we repeat this construction, namely we keep adding beam splitters that each take the output state of the preceding one as their inputs.

g) Calculate the detection probabilities of the output states of such an interferometer with N beam splitters, where $N \in \mathbb{N}$.

h) Imagine that someone performs a measurement on mode B' but does not tell us whether they detected a photon or not. How can we describe the state of the two modes?