SCHMIDT DECOMPOSITION

• $|\psi\rangle \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$

CLAIM: $\exists\, |u_i\rangle \in \mathbb{C}^{d_A},\ |v_i\rangle \in \mathbb{C}^{d_B}\ \forall i \in \{1, \dots, \min(d_A, d_B)\} : \quad |\psi\rangle = \sum_{i=1}^{\min(d_A,d_B)} \lambda_i\, |u_i\rangle \otimes |v_i\rangle$

PROOF:

$$|\psi\rangle = \mathrm{vec}(A) \overset{\text{SVD}}{=} \mathrm{vec}\left(\sum_{i=1}^{\min(d_A,d_B)} \lambda_i\, |u_i\rangle\langle v_i| \right) = \sum_{i=1}^{\min(d_A,d_B)} \lambda_i\, |u_i\rangle \otimes |v_i\rangle$$

$\forall |\psi\rangle : \exists A \in \mathcal{L}(\mathcal{H}_A, \mathcal{H}_B)$

$$\left(\begin{array}{l} |\psi\rangle = \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} \langle i,j|\psi\rangle\, |i\rangle \otimes |j\rangle \\ A = \mathrm{vec}^{-1}(|\psi\rangle) = \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} \langle i,j|\psi\rangle\, |i\rangle\langle j| \end{array} \right)$$

SUPEROPERATOR DECOMPOSITION

• $\Phi : \mathcal{L}(\mathbb{C}^d) \to \mathcal{L}(\mathbb{C}^d)$ linear map.

CLAIM: $\exists\, A_i, B_i \in \mathcal{L}(\mathbb{C}^d)\ \forall i \in \{1, \dots, d^2\} : \quad \Phi(X) = \sum_{i=1}^{d^2} A_i X B_i^\dagger$

PROOF:

• $\rho_\Phi := \Phi \otimes \mathcal{I}\left(|\Omega\rangle\langle\Omega|\right)$ <u>CHOI-STATE</u>

(Id. channel)

$|\Omega\rangle = \sum_{i=1}^{d} |i,i\rangle$

$$\rho_\Phi = \sum_{i=1}^{d^2} \lambda_i\, |u_i\rangle\langle v_i| = \sum_{i=1}^{d^2} \lambda_i\, \mathrm{vec}(\tilde{A}_i) \left(\mathrm{vec}(\tilde{B}_i)\right)^\dagger = \sum_{i=1}^{d^2} \lambda_i\, \tilde{A}_i \otimes \mathbb{1}\, |\Omega\rangle\langle\Omega|\, \tilde{B}_i^\dagger \otimes \mathbb{1}$$

<u>SVD.</u>

• VEC. $\left(\begin{array}{l} \exists\, \tilde{A}_i \in \mathcal{L}(\mathbb{C}^d) : |u_i\rangle = \mathrm{vec}(\tilde{A}_i) \\ \exists\, \tilde{B}_i \in \mathcal{L}(\mathbb{C}^d) : |v_i\rangle = \mathrm{vec}(\tilde{B}_i) \end{array} \right)$

$\mathrm{vec}(\tilde{A}_i) = \tilde{A}_i \otimes \mathbb{1}\, |\Omega\rangle$

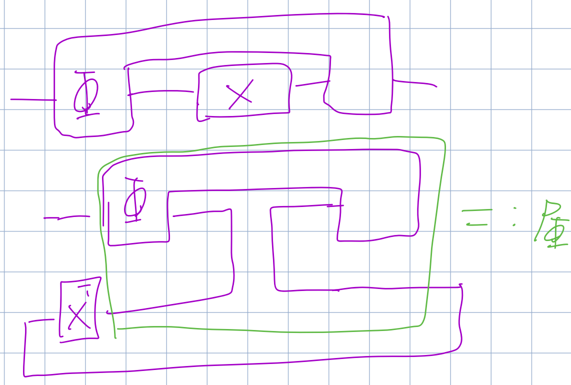
$$\Phi(X) = \mathrm{tr}_B\left(\mathbb{1} \otimes X^T \rho_\Phi \right) = \sum_{i=1}^{d^2} \lambda_i\, \tilde{A}_i X \tilde{B}_i^\dagger \,.$$

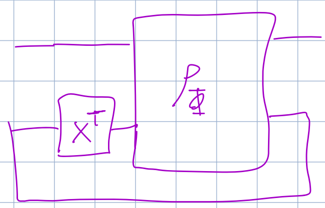
CHOI ISOMORF.

$\rho_\Phi = \sum_{i=1}^{d^2} \lambda_i\, \tilde{A}_i \otimes \mathbb{1}\, |\Omega\rangle\langle\Omega|\, \tilde{B}_i^\dagger \otimes \mathbb{1}$

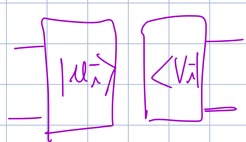
$A_i := \sqrt{\lambda_i}\, \tilde{A}_i$
$B_i := \sqrt{\lambda_i}\, \tilde{B}_i$ $\Rightarrow \blacksquare$

TENSOR NETWORK PROOF:

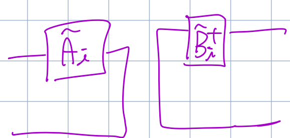
- $\Phi(X) =$ 

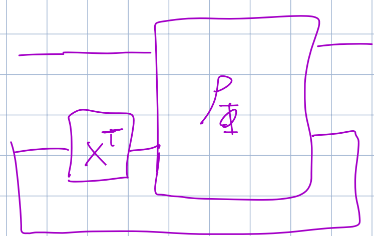
$=$ 

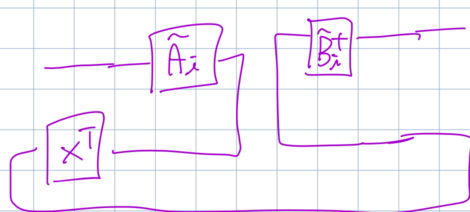
← CHOI - ISOMORPHISM

- $P_\Phi = \sum_{i=1}^{d^2} \lambda_i$ 

↑ SVD

$= \sum_{i=1}^{d^2} \lambda_i$ 

- $\Phi(X) =$ 

$= \sum_{i=1}^{d^2} \lambda_i$ 

$= \sum_{i=1}^{d^2} \lambda_i \tilde{A}_i X \tilde{B}_i^{\dagger}$

KRAUS DECOMPOSITION

- $\Phi : \mathcal{L}(\mathbb{C}^d) \to \mathcal{L}(\mathbb{C}^d)$ linear map.
- Φ COMPLETELY POSITIVE (CP)

CLAIM: $\exists K_i \in \mathcal{L}(\mathbb{C}^d)\ \forall i \in \{1,\dots,d^2\} : \quad \Phi(X) = \sum_{i=1}^{d^2} K_i X K_i^\dagger$

PROOF:

- Same as before but with EIG.DEC. on the Choi state. (instead of SVD).

- In fact $\rho_\Phi := \Phi \otimes \mathbb{I}_d (|\Omega\rangle\langle\Omega|) \underset{\text{CP}}{\geq} 0$

$$\Rightarrow \quad \rho_\Phi = \rho_\Phi^\dagger$$

$$\Rightarrow \quad \rho_\Phi = \sum_{i=1}^{d^2} \lambda_i |u_i\rangle\langle u_i| .$$ (Conclude as before)

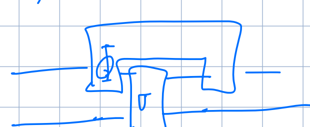
- $\Phi : \mathcal{L}(\mathbb{C}^d) \to \mathcal{L}(\mathbb{C}^d)$ linear map.

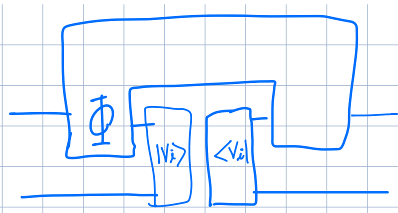
CLAIM: $\underset{\text{CHOI STATE}}{\rho_\Phi} := \Phi \otimes \mathbb{1} (|\Omega\rangle\langle\Omega|) \geq 0 \iff \Phi$ Completely Positive (C.P.)

PROOF:

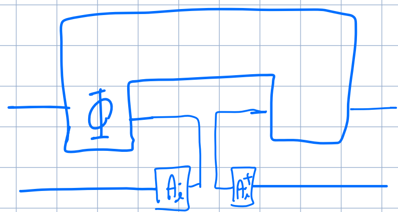
($\Leftarrow$): Φ C.P. $\Rightarrow \left(\forall\ \sigma \geq 0 \Rightarrow \Phi \otimes \mathbb{1}(\sigma) \geq 0 \right) \Rightarrow \rho_\Phi \geq 0$.

($\Rightarrow$): Let $\sigma \in \mathcal{L}(\mathbb{C}^d \otimes \mathbb{C}^d)$: $\sigma \geq 0$.

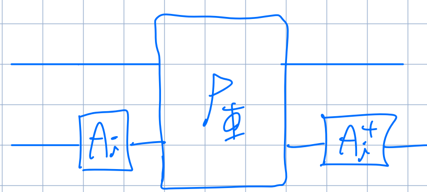
$\Phi \otimes \mathbb{1}(\sigma) =$ 

$= \sum_i \lambda_i$ 

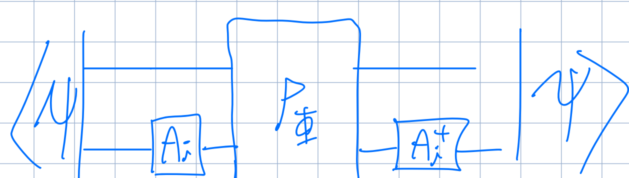
$(\sigma \geq 0 \Rightarrow \sigma^\dagger = \sigma)$

$= \sum_i$ 

$A_i^T := \sqrt{\lambda_i}\, \mathrm{vec}^{-1}(|v_i\rangle)$

$= \sum_i$ 

- Given $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^{d_B}$,

$\langle\psi|\, \Phi \otimes I(\sigma)\,|\psi\rangle = \sum_i$ 

≥ 0