- DEF. VON NEUMAN ENTROPY

$$S(\rho) := -\mathrm{tr}(\rho \log \rho) \quad \text{where } \rho \text{ is a density matrix}$$

- $\underset{\text{EIGENDEC.}}{\rho} = \sum_{i=1}^{d} \underset{\substack{\text{eigenvalues} \\ (\text{with } \sum_{i=1}^{d} p_i = 1)}}{p_i} |v_i\rangle\langle v_i| \Longrightarrow S(\rho) = -\sum_{i=1}^{d} p_i \log(p_i) = \underset{\substack{\text{SHANNON} \\ \text{ENTROPY}}}{H}(\{p_i\})$

PROOF:

In general we have $f(\rho) = \sum_{i=1}^{d} f(p_i) |v_i\rangle\langle v_i|$.

FACT (*):

This because every $f(\cdot)$ can be written as $f(\rho) = \sum_{j=0}^{\infty} c_j \rho^j$

$$\Rightarrow f(\rho) = \sum_{j=0}^{\infty} c_j \rho^j \underset{\sum_{i=1}^{d} |v_i\rangle\langle v_i|}{\mathbb{1}} = \sum_i \sum_j c_j \underbrace{(\rho^j |v_i\rangle)}_{p_i^j |v_i\rangle} \langle v_i| =$$

$$= \sum_i \left(\sum_j c_j p_i^j\right) |v_i\rangle\langle v_i| \underset{\sum_j c_j p_i^j = f(p_i)}{=} \sum_{i=1}^{d} f(p_i) |v_i\rangle\langle v_i|$$

- So given $f(\rho) := \rho \log(\rho) \Rightarrow \rho \log(\rho) = \sum_{i=1}^{d} (p_i \log(p_i)) |v_i\rangle\langle v_i|$

$$\Rightarrow S(\rho) := -\mathrm{tr}(\rho \log \rho) = -\sum_{i=1}^{d} (p_i \log(p_i)) \underbrace{\mathrm{tr}(|v_i\rangle\langle v_i|)}_{1}$$

$$= -\sum_{i=1}^{d} p_i \log p_i$$

OBS.

$$0 \leq S(\rho) \leq \log(d)$$

PROOF:

$S(\rho) = H(\{p_i\})$ and we showed that $0 \leq H(x) \leq \log(d)$.

OBS

- $S(\rho) = 0 \iff \rho = |\psi\rangle\langle\psi|$ (PURE)
- $S(\rho) = \log(d) \iff \rho = \frac{\mathbb{1}}{d}$ (MAXIMALLY MIXED STATE)

PROOF:

- $S(\rho) = H(\{p_i\}) = 0 \underset{\substack{\uparrow \\ \text{PREVIOUS LECTURE}}}{\iff} \{p_i\}$ is a peaked distribution i.e. $p_i = \delta_{i,a}$ $\forall i$.
- $S(\rho) = H(\{p_i\}) = \log(d) \iff \{p_i\}$ is a flat distribution i.e. $p_i = \frac{1}{d}$ $\forall i$.

$$\iff \rho = \sum_{i=1}^{d} p_i |v_i\rangle\langle v_i| = \frac{1}{d} \underbrace{\sum_{i=1}^{d} |v_i\rangle\langle v_i|}_{\mathbb{1}} \Rightarrow \square$$

- $S(\rho) = S(U\rho U^\dagger)$ $\forall U$ unitary.

PROOF:

$\rho = \sum_i p_i |v_i\rangle\langle v_i|$ and $U\rho U^\dagger = \sum_i p_i \left(U|v_i\rangle\right)\left(\langle v_i|U^\dagger\right)$ have the same eigenvalues.
and $S(\cdot)$ depends only by the eigenvalues.

- DEF: (RELATIVE ENTROPY)

$$S(\rho_1 \| \rho_2) := \mathrm{tr}[\rho_1 \log(\rho_1)] - \mathrm{tr}[\rho_1 \log(\rho_2)]$$
$$= -S(\rho_1) - \mathrm{tr}[\rho_1 \log(\rho_2)]$$

TEO ① : (KLEIN INEQUALITY)

$$S(\rho_1 \| \rho_2) \geq 0 \qquad \left(S(\rho_1 \| \rho_2) = 0 \iff \rho_1 = \rho_2 \right)$$

PROOF:

$$\rho_1 = \sum_{i=1}^{d} p_i |v_i\rangle\langle v_i| \quad , \quad \rho_2 = \sum_{j=1}^{d} q_j |w_j\rangle\langle w_j|$$

$$S(\rho_1 \| \rho_2) = -S(\rho_1) - tr\left[\rho_1 \log(\rho_2)\right]$$

$$= -S(\rho_1) - tr\left[\sum_{i=1}^{d} p_i |v_i\rangle\langle v_i| \log(\rho_2)\right]$$

$$= -S(\rho_1) - \sum_{i=1}^{d} p_i \langle v_i| \log(\rho_2) |v_i\rangle$$

$$\overset{\circledast}{=} -S(\rho_1) - \sum_{i=1}^{d} \sum_{j=1}^{d} p_i \log(q_j) \underbrace{\langle v_i|w_j\rangle\langle w_j|v_i\rangle}_{= |\langle v_i|w_j\rangle|^2} =$$

$\circledast$ $\log(\rho_2) = \sum_{j=1}^{d} \log(q_j) |w_j\rangle\langle w_j|$

$$= -S(\rho_1) - \sum_{i=1}^{d} \sum_{j=1}^{d} p_i \log(q_j) |\langle v_i|w_j\rangle|^2$$

$$= +\sum_{i=1}^{d} p_i \log(p_i) - \sum_{i=1}^{d} \sum_{j=1}^{d} p_i \log(q_j) |\langle v_i|w_j\rangle|^2$$

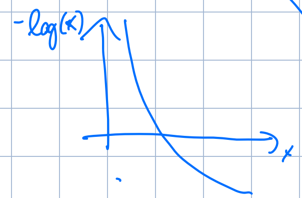
$$= +\sum_i \sum_j |\langle v_i|w_j\rangle|^2 p_i \log(p_i) - \sum_{i=1}^{d} \sum_j |\langle v_i|w_j\rangle|^2 p_i \log(q_j)$$

$\sum_j |\langle v_i|w_j\rangle|^2 = 1$

$$= +\sum_i \sum_j |\langle v_i|w_j\rangle|^2 p_i \left(\log\left(\frac{p_i}{q_j}\right)\right) = \sum_i \sum_j \underbrace{|\langle v_i|w_j\rangle|^2}_{= P(i,j)} p_i \left(-\log\left(\frac{q_j}{p_i}\right)\right)$$

$$\geq \sum_i p_i \left[-\log\left(\sum_j P(i,j)\, \frac{q_j}{p_i}\right)\right] = \sum_i p_i \left[-\log\left(\frac{r_i}{p_i}\right)\right]$$

- $\sum_j P(i,j) = 1$, $P(i,j) \geq 0$
- $-\log(x)$ CONVEX



- $r_i := \sum_j P(i,j)\, q_j$ Prob. dist.
  - $\sum_i r_i = 1$
  - $r_i \geq 0$

$$= \sum_i p_i \log\left(\frac{p_i}{r_i}\right) \geq 0$$

previous lecture

- The first inequality is saturated when $p_i = q_j \ \forall i,j$

$$\Updownarrow$$

$$p_i = q_j = \frac{1}{d} \quad \forall i,j$$

or when $P(i,j) = \delta_{j,i_0}$. $\forall j = 1, \dots, d$

- But $P(i,j) = |\langle v_i | w_j \rangle|^2 = \delta_{j,i_0} \ \forall j = 1,\dots,d \iff |v_i\rangle = |w_{i_0}\rangle \ \forall i$

$$|v_i\rangle = \sum_j \underbrace{\langle w_j | v_i \rangle}_{=\delta_{j,i_0} e^{i\phi(i_0)}} |w_j\rangle = e^{i\phi(i_0)} |w_{i_0}\rangle$$

- The second inequality is saturated when $p_i = r_i \ \forall i$ (previous lecture)

$$\iff p_i = \sum_j P(i,j)\, q_j = \sum_j \delta_{j,i_0}\, q_j = q_{i_0} \quad \forall i$$

$$\iff \rho_1 = \sum_i p_i |v_i\rangle\langle v_i| = \sum_i q_{i_0} |w_{i_0}\rangle\langle w_{i_0}| = \rho_2$$

• OBS.

- $S(\rho_1 \| \rho_2) \neq S(\rho_2 \| \rho_1)$ (NOT SYMMETRIC)

- $\mathrm{Supp}(\rho_1) \cap \mathrm{Ker}(\rho_2) \neq \{\vec{0}\} \Rightarrow S(\rho_1 \| \rho_2) = +\infty$

PROOF:

$$S(\rho_1 \| \rho_2) = -S(\rho_1) - \sum_{i=1}^{d} \sum_{j=1}^{d} p_i \log(q_j) |\langle v_i | w_j \rangle|^2$$

If $q_j = 0$ but $p_i |\langle v_i | w_j \rangle| \neq 0 \ \forall i \Longrightarrow S(\rho_1 \| \rho_2) = +\infty$

$\Updownarrow$ $|w_j\rangle \in \mathrm{Ker}(\rho_2)$ $\quad \Updownarrow$ $|w_j\rangle \notin \mathrm{Ker}^{\perp}(\rho_1) = \mathrm{supp}(\rho_1)$

- So $\mathrm{Ker}(\rho_2) \cap \mathrm{supp}(\rho_1) \neq \{0\} \Rightarrow S(\rho_1 \| \rho_2) = +\infty \Rightarrow \square$

- $O = O_1 \otimes O_2$ with $O_1 = O_1^{\dagger}$, $O_2 = O_2^{\dagger}$

$$\log(O) = \log(O_1) \otimes \mathbb{1} + \mathbb{1} \otimes \log(O_2) = ⍟$$

PROOF:

$$\log(O) = \log(O_1 \otimes O_2) \underset{\uparrow}{=} \log\left[\sum_{i,j} \lambda_i^{(1)} \lambda_j^{(2)} \left(|v_i^{(1)}\rangle\langle v_i^{(1)}| \otimes |v_i^{(2)}\rangle\langle v_i^{(2)}|\right)\right] =$$

$O_1 = \sum_i \lambda_i^{(1)} |v_i^{(1)}\rangle\langle v_i^{(1)}|$

$O_2 = \sum_j \lambda_j^{(2)} |v_j^{(2)}\rangle\langle v_j^{(2)}|$

eigenvectors for $O_1 \otimes O_2$

$$\underset{\uparrow}{=} \sum_{i,j} \log\left(\lambda_i^{(1)} \lambda_j^{(2)}\right) \left(|v_i^{(1)}\rangle\langle v_i^{(1)}| \otimes |v_i^{(2)}\rangle\langle v_i^{(2)}|\right) \underset{\uparrow}{=}$$

$f(A) = \sum_i f(\lambda_i) |v_i\rangle\langle v_i|$, $A = \sum \lambda_i |v_i\rangle\langle v_i|$

$\log(ab) = \log(a) + \log(b)$

$$= \sum_{i,j} \log(\lambda_i^{(1)}) \left(|v_i^{(1)}\rangle\langle v_i^{(1)}| \otimes |v_i^{(2)}\rangle\langle v_i^{(2)}|\right) + \sum_{i,j} \log(\lambda_j^{(2)}) \left(|v_i^{(1)}\rangle\langle v_i^{(1)}| \otimes |v_i^{(2)}\rangle\langle v_i^{(2)}|\right)$$

$$= \sum_i \log(\lambda_i^{(1)}) \left(|v_i^{(1)}\rangle\langle v_i^{(1)}| \otimes \mathbb{1}\right) + \sum_j \log(\lambda_j^{(2)}) \left(\mathbb{1} \otimes |v_i^{(2)}\rangle\langle v_i^{(2)}|\right) =$$

$$= \log(O_1) \otimes \mathbb{1} + \log(O_2) \otimes \mathbb{1}$$

COR.

- $O = O_1 \otimes O_2 \otimes \cdots \otimes O_n \quad , O_i = O_i^{\dagger} \quad \forall i$
- $\log(O) = \sum_{i=1}^{N} \mathbb{1}_{1,\dots,i-1} \otimes \log(O_i) \otimes \mathbb{1}_{i+1,\dots,n}$

COR:

$$S(\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_N) = \sum_{i=1}^{N} S(\rho_i)$$

PROOF:

$$\log(\rho) = \sum_{i=1}^{N} \mathbb{1}_{1,\dots,i-1} \otimes \log(\rho_i) \otimes \mathbb{1}_{i+1,\dots,n}$$

$$S(\rho) = -tr\left(\rho \log(\rho)\right) = -\sum_{i=1}^{N} tr\left(\rho\left(\mathbb{1}_{1,\dots,i-1} \otimes \log(\rho_i) \otimes \mathbb{1}_{i+1,\dots,n}\right)\right) =$$

$$= -\sum_{i=1}^{N} tr\left(\underbrace{tr_{\bar{i}}(\rho)}_{= \rho_i} \log(\rho_i)\right) = \sum_{i=1}^{N} S(\rho_i)$$

THEOR. ②

- $\rho_A = tr_B(\rho_{AB}) \quad , \rho_B = tr_A(\rho_{AB})$
- $S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B) \quad \left(S(\rho_{AB}) = S(\rho_A) + S(\rho_B) \iff \rho_{AB} = \rho_A \otimes \rho_B\right)$

PROOF:

- $S(\rho_{AB} \| \rho_A \otimes \rho_B) \geq 0$

  $\uparrow$ KLEIN INEQUALITY

- $\underbrace{S(\rho_{AB} \| \rho_A \otimes \rho_B)}_{\geq 0} = -S(\rho_{AB}) - tr\left(\rho_{AB} \underbrace{\log(\rho_A \otimes \rho_B)}_{= \log(\rho_A) \otimes \mathbb{1} + \mathbb{1} \otimes \log(\rho_B)}\right)$

  $\Leftarrow$ ⊛

$$= -S(\rho_{AB}) - \underbrace{tr\left(\rho_{AB} \log(\rho_A) \otimes \mathbb{1}\right)}_{tr\left(\rho_A \log(\rho_A)\right)} - tr\left(\rho_{AB}\, \mathbb{1} \otimes \log(\rho_B)\right)$$

$$= -S(\rho_{AB}) + S(\rho_A) + S(\rho_B) \geq 0 \quad \Rightarrow \quad \underline{\square}$$

- "=" $\Leftrightarrow$ $S(\rho_{AB} \| \rho_A \otimes \rho_B) = 0 \Leftrightarrow \rho_{AB} = \rho_A \otimes \rho_B$

FACT ③ :

- $\rho_A := tr_B(\rho)$, $\rho_B := tr_A(\rho)$
- $\rho = |\psi\rangle\langle\psi|$ pure $\Rightarrow$ $S(\rho_A) = S(\rho_B)$

(This is not true in general if ρ is not pure)
Find a counter-example ☺.

PROOF:

SCHMIDT.

$$|\psi\rangle = \sum_{i=1}^{\min(d_A, d_B)} \sqrt{\lambda_i}\, |v^A_i\rangle \otimes |v^B_i\rangle \qquad \text{with } \langle v^A_i, v^A_j\rangle = \delta_{i,j}, \ \langle v^B_i, v^B_j\rangle = \delta_{i,j}.$$

SUBPROOF:

- $$|\psi\rangle = \sum_{i,j} \underbrace{\langle i,j|\psi\rangle}_{(C)_{i,j}} |i,j\rangle \underset{\text{SVD}}{=} \sum_{i,j} (UDV^\dagger)_{i,j}\, |i,j\rangle =$$

$\bullet\ C = UDV^\dagger$

$$\underset{\langle i|U|l\rangle = U_{i,l}}{=} \sum_l \sum_{i,j} U_{i,l} D_{ll} (V^\dagger)_{l,j} |i\rangle \otimes |j\rangle \underset{\sqrt{\lambda_l} := D_{ll} \geq 0}{=} \sum_l \lambda_l \underbrace{\left(\sum_i U_{i,l}|i\rangle\right)}_{U|l\rangle = |v^A_i\rangle} \otimes \underbrace{\left[\sum_j (V^\dagger)_{l,j}|j\rangle\right]}_{V^*|l\rangle = |v^B_i\rangle}$$

$$tr_B(|\psi\rangle\langle\psi|) = \sum_i \lambda_i\, |v_i^{(A)}\rangle\langle v_i^{(A)}|$$

$$tr_A(|\psi\rangle\langle\psi|) = \sum_i \lambda_i\, |v_i^{(B)}\rangle\langle v_i^{(B)}|$$

$\Rightarrow$ Same eigenvalues $\Rightarrow$ same entropies.

THEOR ④

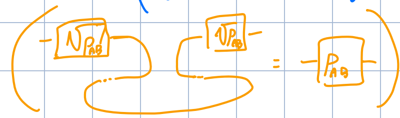
$$S(\rho_{AB}) \geq |S(\rho_A) - S(\rho_B)|$$

PROOF:

- Given a mixed state ρ_{AB}, then $\exists$ R such that $\rho_{AB} = tr_R[\underbrace{|\Psi\rangle\langle\Psi|}_{=\rho_{ABR}}]$ with $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_R$

SUBPROOF: (ALREADY SEEN)

(E.g. take $|\Psi\rangle := \sqrt{\rho_{AB}} \otimes \mathbb{1}_R \left(\sum_{i=1}^{d} |i\rangle_{AB} \otimes |i\rangle_R \right)$ and verify.


)

- We have:

- $$S(\rho_B) \underset{\uparrow}{=} S(\rho_{AR}) \underset{\uparrow (\text{TH}②)}{\leq} S(\rho_A) + S(\rho_R) \underset{\uparrow}{=} S(\rho_A) + S(\rho_{AB})$$

[• $S(\rho_B) \underset{\uparrow}{=} S(\rho_{AR})$ (• FACT ③ • ρ_{ABR} pure)]

[• $S(\rho_{AB}) \underset{\uparrow}{=} S(\rho_R)$ (• FACT ③ • ρ_{ABR} pure)]

$$\Longrightarrow \boxed{S(\rho_{AB}) \geq S(\rho_B) - S(\rho_A)} \equiv \underline{*_1}$$

- $$S(\rho_A) \underset{\uparrow \text{FACT}③}{=} S(\rho_{BR}) \underset{\uparrow \text{TH}②}{\leq} S(\rho_B) + S(\rho_R) \underset{\uparrow \text{FACT}③}{=} S(\rho_B) + S(\rho_{AB})$$

$$\Longrightarrow \boxed{S(\rho_{AB}) \geq S(\rho_A) - S(\rho_B)} \equiv \underline{*_2}$$

$$\underset{\uparrow *_1, *_2}{\Longrightarrow} \square$$

OBS| Also in the classical case I have $H(X,Y) \geq |H(X) - H(Y)|$ but we did not mention it because in the classical case we have an even stronger inequality: $H(X,Y) \geq \mathrm{MAX}(H(X), H(Y))$ which can be violated quantumly.

$$\boxed{\underset{\text{FALSE}}{S(\rho_{AB})} \geq \max(S(\rho_A), S(\rho_B))}$$

NO! FALSE! NO! NO!

PROOF.

COUNTER-EXAMPLE: $\rho_{AB} = |\psi\rangle\langle\psi|$ with $|\psi\rangle = |\Omega\rangle$ max. entangled state.

- $S(\rho_{AB}) = 0$ ← state pure.
- $\max(S(\rho_A), S(\rho_B)) = \max\left(S\left(\underbrace{\frac{\mathbb{1}}{d_A}}_{\#0}\right), S\left(\underbrace{\frac{\mathbb{1}}{d_B}}_{\#0}\right)\right) \neq 0$

DEF|.

$$\rho \text{ Separable} \iff \rho = \sum_{i=1}^{N} p_i\, \rho_1^{(i)} \otimes \rho_2^{(i)} \quad \text{with} \begin{cases} p_i \geq 0 \\ \sum_{i=1}^{N} p_i = 1 \end{cases}$$

$\rho_1^{(i)}$ and $\rho_2^{(i)}$ states.

DEF|

$$\rho \text{ entangled} \iff \rho \text{ Not separable.}$$

FACT 1:

$$\rho \text{ separable} \implies S(\rho_{AB}) \geq S(\rho_A)$$

PROOF: NOT SHOWN HERE.

THEOR. (CONCAVITY)

$$S\left(\sum_{i=1}^{N} p_i \rho_i\right) \geq \sum_{i=1}^{N} p_i S(\rho_i) \qquad \left(S\left(\sum_{i=1}^{N} p_i \rho_i\right) = \sum_{i=1}^{N} p_i S(\rho_i) \Leftrightarrow \rho_i = \rho \ \forall i \right)$$

PROOF:

- $$\rho_{AB} := \sum_{i=1}^{N} p_i \rho_i \otimes |i\rangle\langle i| \underset{\text{EIGEN.}}{=} \sum_{i=1}^{N} \sum_{j=1}^{d} p_i \lambda_j^{(i)} |v_j^{(i)}\rangle\langle v_j^{(i)}| \otimes |i\rangle\langle i|$$

$$S(\rho_{AB}) = H(\{p_i \cdot \lambda_j^{(i)}\}) = -\sum_{i,j} p_i \lambda_j^{(i)} \log(p_i \lambda_j^{(i)}) = -\sum_{i,j} p_i \lambda_j^{(i)} \log(p_i) - \sum_{i,j} p_i \lambda_j^{(i)} \log(\lambda_j^{(i)})$$

$$\underset{\left(\sum_j \lambda_j^{(i)} = 1\right)}{=} H(\{p_i\}) + \sum_{i=1}^{N} p_i S(\rho_i)$$

- $$\rho_A = \mathrm{tr}_B(\rho_{AB}) = \sum_{i=1}^{N} p_i \rho_i$$

$$S(\rho_A) = S\left(\sum_{i=1}^{N} p_i \rho_i\right)$$

- $$\rho_B = \mathrm{tr}_A(\rho_{AB}) = \sum_{i=1}^{N} p_i |i\rangle\langle i|$$

$$S(\rho_B) = H(\{p_i\})$$

- $$S(\rho_{AB}) \underset{\text{SUB-ADD.}}{\leq} S(\rho_A) + S(\rho_B)$$

$$\| $$

$$H(\{p_i\}) + \sum_{i=1}^{N} p_i S(\rho_i) \leq S\left(\sum_{i=1}^{N} p_i \rho_i\right) + H(\{p_i\}) \Rightarrow \square$$

THEOR.

$$\sum_{i=1}^{N} p_i S(\rho_i) \leq S\left(\sum_{i=1}^{N} p_i \rho_i\right) \leq H(\{p_i\}) + \sum_{i=1}^{N} p_i S(\rho_i)$$

PROOF:

- $$S\left(\sum_{i=1}^{N} p_i \rho_i\right) \underset{\rho_i = \sum_{i=1}^{d} \lambda_j |v_j\rangle\langle v_j|}{=} S\left(\sum_{i=1}^{N} \sum_{j=1}^{d} p_i \lambda_j^{(i)} |v_j^{(i)}\rangle\langle v_j^{(i)}|\right)$$

ortonorm. basis

- $$|\psi\rangle_{AB} := \sum_{i=1}^{N} \sum_{j=1}^{d} \sqrt{p_i \lambda_j^{(i)}}\, |v_j^{(i)}\rangle \otimes |(i,j)\rangle$$

- $$S(\rho_A) = S\left(tr_B\left[|\psi\rangle\langle\psi|\right]\right) \underset{|\psi\rangle\langle\psi| \text{ pure}}{=} S\left(tr_A\left[|\psi\rangle\langle\psi|\right]\right) = S(\rho_B)$$

$$\| $$

$$S\left(\sum_{i,j} p_i \lambda_j^{(i)} |v_j^{(i)}\rangle\langle v_j^{(i)}|\right)$$

$$\| $$

$$S\left(\sum_{i=1}^{N} p_i \rho_i\right)$$

$$\Rightarrow S(\rho_B) = S\left(\sum_{i=1}^{N} p_i \rho_i\right)$$

- $$\rho_B = tr_A\left[|\psi\rangle\langle\psi|\right] = tr_A\left(\sum_{i=1}^{N}\sum_{j=1}^{d}\sum_{l=1}^{N}\sum_{k=1}^{d} \sqrt{p_i \lambda_j^{(i)}}\sqrt{p_l \lambda_k^{(l)}}\, |v_j^{(i)}\rangle\langle v_k^{(l)}| \otimes |(i,j)\rangle\langle(l,k)|\right) =$$

$$= \sum_{m=1}^{d} \langle m| \otimes \mathbb{1} \left(\sum_{i=1}^{N}\sum_{j=1}^{d}\sum_{l=1}^{N}\sum_{k=1}^{d} \sqrt{p_i \lambda_j^{(i)}}\sqrt{p_l \lambda_k^{(l)}}\, |v_j^{(i)}\rangle\langle v_k^{(l)}| \otimes |(i,j)\rangle\langle(l,k)|\right) |m\rangle \otimes \mathbb{1}$$

$$= \sum_{m=1}^{d} \sum_{i=1}^{N}\sum_{j=1}^{d}\sum_{l=1}^{N}\sum_{k=1}^{d} \sqrt{p_i \lambda_j^{(i)}}\sqrt{p_l \lambda_k^{(l)}}\, \langle m|v_j^{(i)}\rangle\langle v_k^{(l)}|m\rangle\, |(i,j)\rangle\langle(l,k)|$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{d}\sum_{l=1}^{N}\sum_{k=1}^{d} \sqrt{p_i \lambda_j^{(i)}}\sqrt{p_l \lambda_k^{(l)}}\, \langle v_k^{(l)}|v_j^{(i)}\rangle\, |(i,j)\rangle\langle(l,k)|$$

- If I perform a measurement on B (with PVM $\{\underbrace{|(i,j)\rangle\langle(i,j)|}_{P_{i,j}}\}_{i,j}$), then the post measurement state will be:

$$\rho_B' = \sum_{i'=1}^{d}\sum_{j'=1}^{N} tr\left(\rho_B P_{i',j'}\right)\left(\frac{P_{i',j'}\,\rho_B\,P_{i',j'}}{tr\left(\rho_B P_{i',j'}\right)}\right)$$

$$= \sum_{i'=1}^{d}\sum_{j'=1}^{N} P_{i',j'}\,\rho_B\,P_{i',j'} = \sum_{i',j'} p_{i'} \lambda_{j'}^{(i')}\, |(i',j')\rangle\langle(i',j')|$$

$$\rho_B' = \sum_{i,s} p_i \lambda_s^{(i)} \, |(i,s)\rangle\langle(i,s)|$$

$$S(\rho_B') = H\left(\{p_i \lambda_s^{(i)}\}\right) = H\left(\{p_i\}\right) + \sum_i p_i \underbrace{\sum_{s=1}^{d} \lambda_s^{(i)} \log\left(\lambda_s^{(i)}\right)}_{S(\rho_i)}$$

• $\boxed{\text{LEMMA}}$ • $\{P_K\}_K$ PVM set $\Rightarrow$ $S\Big(\underbrace{\sum_K P_K \rho P_K}_{\rho'}\Big) \geq S(\rho)$

(PROOF LATER)

ρ' ← state after measurement.

• $$S(\rho_B') \geq S(\rho_B) = S\left(\sum_i p_i \rho_i\right)$$

$$S(\rho_B') = H\left(\{p_i\}\right) + \sum_i p_i S(\rho_i)$$

• $\boxed{\text{LEMMA}}$ • $\{P_K\}_K$ PVM set $\Rightarrow$ $S\Big(\underbrace{\sum_K P_K \rho P_K}_{\rho'}\Big) \geq S(\rho)$

ρ' ← state after measurement.

SUB PROOF:

• $$0 \underset{\text{KLEIN INEQ}}{\leq} S(\rho \| \rho') = -S(\rho) - \mathrm{tr}(\rho \log(\rho'))$$

• $$-\mathrm{tr}(\rho \log(\rho')) \geq S(\rho)$$

- $-\mathrm{tr}\left(\rho \log(\rho')\right) = -\mathrm{tr}\left(\rho' \log(\rho')\right) \equiv S(\rho') \Rightarrow \square$

SUB SUB PROOF:

- $\mathrm{tr}\left(\rho \log(\rho')\right) \underset{\uparrow}{=} \mathrm{tr}\left(\sum_i P_i \, \rho \log(\rho')\right) = \sum_i \mathrm{tr}\left(P_i^2 \rho \log(\rho')\right) \underset{\uparrow}{=} \sum_i \mathrm{tr}\left(P_i \rho \log(\rho') P_i\right) =$

$\left(\text{PVM} \Rightarrow \cdot\, P_i \cdot P_j = \delta_{i,j} P_i \quad \cdot \sum_i P_i = \mathbb{1}\right)$ $\qquad \mathrm{tr}(AB) = \mathrm{tr}(BA)$

- $\log(\rho') P_i = P_i \log(\rho')$

- $\rho' = \sum_j P_j \rho P_j \Rightarrow \rho' P_i = P_i \rho P_i = P_i \rho' \underset{\uparrow}{\Rightarrow} \log(\rho') P_i = P_i \log(\rho')$

$\log(\rho') = \sum_{k=0}^{\infty} c_k \rho^k$

$= \sum_i \mathrm{tr}\left(P_i \rho P_i \log(\rho')\right) = \sum_i \mathrm{tr}\left(\rho' \log(\rho')\right) \Rightarrow \square$

THEOR. (SSA) (STRONG SUB-ADDITIVITY)

$$S(\rho_{ABC}) + S(\rho_C) \leq S(\rho_{AC}) + S(\rho_{BC})$$

with $\rho_{AB} = \mathrm{tr}_C(\rho_{ABC})$, $\rho_{BC} = \mathrm{tr}_A(\rho_{ABC})$, $\rho_A = \mathrm{tr}_{BC}(\rho_{ABC})$, ...

OBS STRONG SUB-ADDITIVITY $\Rightarrow$ SUB-ADDITIVITY

PROOF:

- $\rho_{ABC} := \rho_{AB} \otimes |0\rangle\langle 0|$

$$\underbrace{S(\rho_{ABC})}_{\substack{\| \\ S(\rho_{AB}) + \underbrace{S(|0\rangle\langle 0|)}_{\substack{\| \\ 0}}}} + \underbrace{S(\rho_C)}_{\substack{\| \\ S(|0\rangle\langle 0|) \\ \| \\ 0}} \leq \underbrace{S(\rho_{AC})}_{\substack{\| \\ S(\rho_A \otimes |0\rangle\langle 0|) \\ \| \\ S(\rho_A) + \underbrace{S(|0\rangle\langle 0|)}_{\substack{\| \\ 0}}}} + \underbrace{S(\rho_{BC})}_{\substack{\| \\ S(\rho_B \otimes |0\rangle\langle 0|) \\ \| \\ S(\rho_B) + \underbrace{S(|0\rangle\langle 0|)}_{\substack{\| \\ 0}}}}$$

PRODUCT STATE $\rightarrow$; $\leftarrow$ PURE

$\Rightarrow S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B) \Rightarrow \square$

PROP.

- $\rho^{\otimes n} := \overbrace{\rho \otimes \cdots \otimes \rho}^{n\text{-times}}$

- $S(\rho^{\otimes n}) \underset{\substack{\uparrow \\ \text{SUB-ADD.} \\ \text{(PRODUCT)}}}{=} n S(\rho)$

- $S(\rho^{\otimes n} \| \sigma^{\otimes n}) = n S(\rho \| \sigma)$

PROOF:

$$S(\rho^{\otimes n} \| \sigma^{\otimes n}) = -S(\rho^{\otimes n}) + \mathrm{tr}\left(\rho^{\otimes n} \log(\sigma^{\otimes n})\right) =$$

$$= -n S(\rho) + n\, \mathrm{tr}(\rho \log(\sigma)) = n S(\rho \| \sigma).$$

$\uparrow$

$\left(\begin{array}{l} \cdot\ S(\rho^{\otimes n}) = n S(\rho) \\ \cdot\ \log(A \otimes B) = \log A \otimes \mathbb{1} + \mathbb{1} \otimes \log(B) \end{array}\right)$