BASICS OF QUANTUM COMPUTING

- $|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

- $|+\rangle := \dfrac{|0\rangle + |1\rangle}{\sqrt{2}}$; $|-\rangle := \dfrac{|0\rangle - |1\rangle}{\sqrt{2}}$; $|\pm_Y\rangle = \dfrac{|0\rangle \pm i\,|1\rangle}{\sqrt{2}}$

- X, Y, Z Pauli : $-\boxed{X}-$, $-\boxed{Y}-$, $-\boxed{Z}-$

- P Pauli $\Rightarrow \mathrm{tr}(P) = 0$, $P = P^{+}$, $P^2 = \mathbb{1}$. $\mathrm{tr}(XY) = 0$, $\mathrm{tr}(YZ) = 0$, $\mathrm{tr}(XZ) = 0$

- $X|+\rangle = +|+\rangle$, $X|-\rangle = -|-\rangle$;
  $Y|\pm_Y\rangle = \pm|\pm_Y\rangle$

- $H|0\rangle = |+\rangle$, $H|1\rangle = |-\rangle \Rightarrow H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $H^2 = \mathbb{1}$, $H^{+} = H$.

- $Z|+\rangle = |-\rangle$, $Z|-\rangle = |+\rangle$

- If $\mathcal{H} = (\mathbb{C}^2)^{\otimes n}$, $n=2 \Rightarrow$

$$|0\rangle_{1,2} = |0\rangle_1 \otimes |0\rangle_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$|1\rangle_{1,2} = |0\rangle_1 \otimes |1\rangle_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|2\rangle_{1,2} = |1\rangle_1 \otimes |0\rangle_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$|3\rangle_{1,2} = |1\rangle_1 \otimes |1\rangle_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

- If $\mathcal{H} = (\mathbb{C}^2)^{\otimes n} \Rightarrow$

$$|0\rangle := |0\rangle_1 \otimes \cdots \otimes |0\rangle_n = |0\rangle^{\otimes N}$$

$$|1\rangle := |0\rangle_1 \otimes \cdots \otimes |1\rangle_n \ ;$$

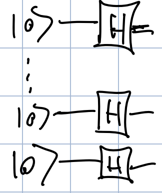
$$|2\rangle := |0\rangle_1 \otimes \cdots \otimes |1\rangle_{n-1} \otimes |0\rangle_n \ ;$$

$$\vdots$$

$$|2^n - 1\rangle := |1\rangle_1 \otimes \cdots \otimes |1\rangle \otimes |1\rangle_n$$

- $H^{\otimes N}|0\rangle^{\otimes N} = (H|0\rangle) \otimes (H|0\rangle) \otimes \cdots \otimes (H|0\rangle) = |+\rangle \otimes \cdots \otimes |+\rangle = |+\rangle^{\otimes N}$

- $|+\rangle^{\otimes N} = \left(\dfrac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes \cdots \otimes \left(\dfrac{|0\rangle + |1\rangle}{\sqrt{2}}\right) = \dfrac{1}{\sqrt{2^n}} \displaystyle\sum_{x_1,\dots,x_n \in \{0,1\}^n} \overbrace{|x_1\rangle \otimes \cdots \otimes |x_n\rangle}^{|x_1,\dots,x_n\rangle}$

- 

$= H^{\otimes N} |0\rangle^{\otimes N} = |+\rangle^{\otimes N}$ NOT ENTANGLED!

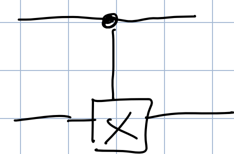
- if $x = 0,1 \Rightarrow H|x\rangle = \frac{|0\rangle + (-1)^x |1\rangle}{\sqrt{2}} = \sum_{z=\{0,1\}} \frac{(-1)^{x \cdot z}}{\sqrt{2}} |z\rangle$

- $H^{\otimes n} |x_1, \dots, x_n\rangle = \sum_{z_1=\{0,1\}} \frac{(-1)^{x_1 z_1}}{\sqrt{2}} |z_1\rangle \otimes \dots \otimes \sum_{z_n=\{0,1\}} \frac{(-1)^{x_n z_n}}{\sqrt{2}} |z_n\rangle$

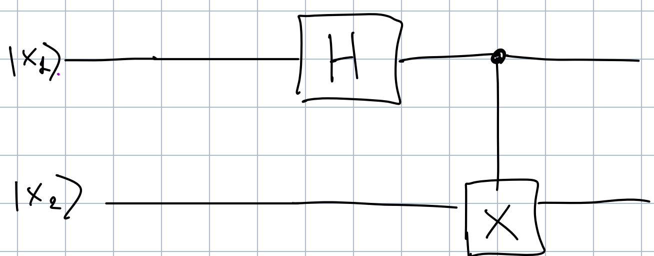
$$= \frac{1}{\sqrt{2^n}} \sum_{z_1,\dots,z_n=\{0,1\}} (-1)^{x_1 z_1 + \dots + x_n z_n} \underline{|z_1,\dots,z_n\rangle}$$

$H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{z \in \{0,1\}^n} (-1)^{x \cdot z} |z\rangle$ with $x \cdot z = x_1 z_1 + \dots + x_n z_n$.

PAULI STRINGS

- $P = P_1 \otimes \dots \otimes P_n$ with $P_i = \{\mathbb{1}, X, Y, Z\} \Rightarrow tr(P) = 0$, $P = P^\dagger$, $P^2 = \mathbb{1}$.
- P^i, P^j Pauli strings $\Rightarrow tr(P_i P_j) = 2^n \delta_{i,j}$.
- $CNOT_{(A,B)} := |0\rangle\langle 0|_A \otimes \mathbb{1} + |1\rangle\langle 1|_B \otimes X_B \equiv$ 

- CNOT generates entanglement:



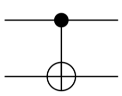
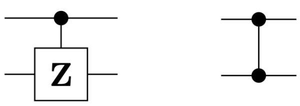
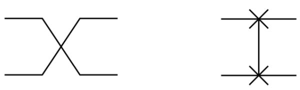
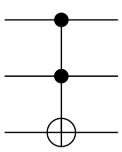
$= CNOT_{(1,2)} (H \otimes \mathbb{1}) |x_1\rangle \otimes |x_2\rangle =$

$= CNOT_{(1,2)} \frac{(|0\rangle + (-1)^{x_1} |1\rangle)}{\sqrt{2}} \otimes |x_2\rangle =$

$= \frac{1}{\sqrt{2}} |0\rangle \otimes |x_2\rangle + \frac{(-1)^{x_1}}{\sqrt{2}} |1\rangle \otimes |\bar{x}_2\rangle$

If $x_1, x_2 = 0 \Rightarrow \frac{|00\rangle + |11\rangle}{\sqrt{2}}$

• OTHER GATES:

Operator	Gate(s)		Matrix
Pauli-X (X)	$\boxed{X}$	$\oplus$	$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = i e^{-i\frac{\pi}{2} X} = \left(\cos\left(\frac{\pi}{2}\right)\mathbb{1} - i \sin\left(\frac{\pi}{2}\right) X\right) i$
Pauli-Y (Y)	$\boxed{Y}$		$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
Pauli-Z (Z)	$\boxed{Z}$		$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = i e^{-i\frac{\pi}{2} Z}$
Hadamard (H)	$\boxed{H}$		$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
Phase (S, P)	$\boxed{S}$		$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{+i\frac{\pi}{2}} \end{pmatrix} = e^{i\frac{\pi}{4}} \begin{pmatrix} e^{-i\frac{\pi}{4}} & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix} = e^{i\frac{\pi}{4}} e^{-i\frac{\pi}{4} Z} = e^{i\frac{\pi}{4}} R_z\left(\frac{\pi}{2}\right)$
$\pi/8$ (T)	$\boxed{T}$		$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$
Controlled Not (CNOT, CX)			$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
Controlled Z (CZ)			$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$
SWAP			$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
Toffoli (CCNOT, CCX, TOFF)			$\begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&1&0 \end{bmatrix}$

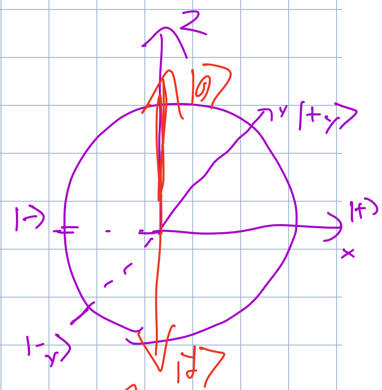
Rotation of π around z in the Bloch sphere.

$P^2 = \mathbb{1} \Rightarrow$

$e^{-i\frac{\theta}{2} P} = \cos\frac{\theta}{2}\, \mathbb{1} - i \sin\frac{\theta}{2}\, P$

$\left(e^{A} = \sum_{k=1}^{\infty} \frac{A^k}{k!}\right)$

$= R_P(\theta)$



action of X.

• $C\text{-}U_{(A,B)} = |0\rangle\langle 0|_A \otimes \mathbb{1}_B + |1\rangle\langle 1|_A \otimes U_B$

• $HZH = X$, $X = HZH$

$\Big(HZH|0\rangle = HZ|+\rangle\ H|-\rangle = |1\rangle = X|0\rangle \Rightarrow HZH = X \Rightarrow Z = HXH$ ($H^2 = \mathbb{1}$)

$HZH|1\rangle = X|1\rangle \Big)$

• $(HS^\dagger)^\dagger Z (HS^\dagger) = Y$ $\quad (SXS^\dagger = Y)$

$$(HS^\dagger)^\dagger Z HS^\dagger = SHZHS^\dagger = SXS^\dagger = Y$$

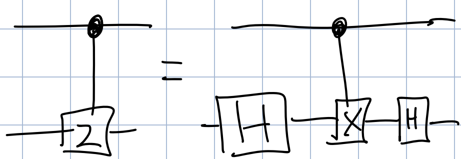
$$SXS^\dagger|0\rangle = i|1\rangle = Y|0\rangle$$

$$SXS^\dagger|1\rangle = -i|0\rangle = Y|1\rangle$$

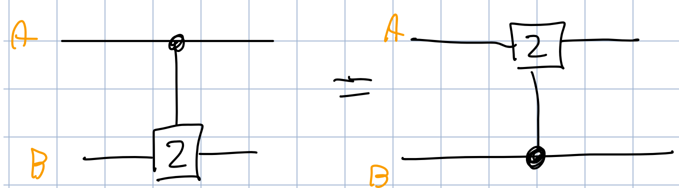
- $$C\text{-}Z = |0\rangle\langle 0| \otimes \overset{HH}{\overset{\shortparallel}{\mathbb{1}}} + |1\rangle\langle 1| \otimes \overset{HXH}{\overset{\shortparallel}{Z}}$$

$$\Rightarrow C\text{-}Z = \mathbb{1}\otimes H\,(C\text{-}NOT)\,\mathbb{1}\otimes H, \qquad C\text{-}NOT = \mathbb{1}\otimes H\,(C\text{-}Z)\,\mathbb{1}\otimes H$$

- $H^2 = \mathbb{1}$
- $HXH = Z$



- $$C\text{-}Z_{(A,B)} = C\text{-}Z_{(B,A)}$$



- They act equally on computational bases:

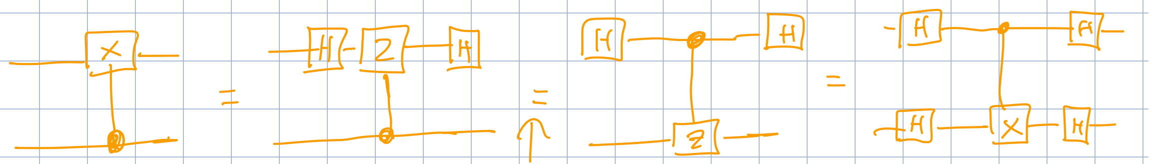
$$C\text{-}Z_{(A,B)}|x_A\rangle\otimes|x_B\rangle = (-1)^{x_A\cdot x_B}|x_A\rangle\otimes|x_B\rangle$$

$$= C\text{-}Z_{(B,A)}|x_A\rangle\otimes|x_B\rangle$$

- $$C\text{-}NOT_{(B,A)} = H^{\otimes 2}\, C\text{-}NOT_{(A,B)}\, H^{\otimes 2}$$

PROOF:

$C\text{-}NOT_{(BA)} =$



$C\text{-}Z_{(A,B)} = C\text{-}Z_{(B,A)}$

$$= H^{\otimes 2}\, C\text{-}NOT_{(B,A)}\, H^{\otimes 2}$$

• $\mathrm{SWAP}\,|i\rangle\otimes|j\rangle := |j\rangle\otimes|i\rangle \qquad \forall\, i,j=0,1$

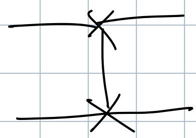
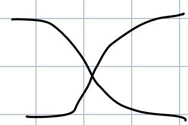
$\mathrm{SWAP}\,|\psi\rangle\otimes|\phi\rangle = |\phi\rangle\otimes|\psi\rangle$

PROOF

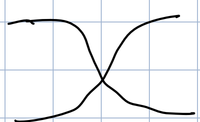
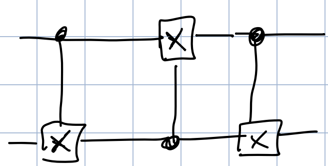
$\mathrm{SWAP}\,|\psi\rangle\otimes|\phi\rangle = \sum_{i,j}\psi_i\phi_j\,\mathrm{SWAP}\,|i\rangle\otimes|j\rangle = \sum_{i,j}\psi_i\phi_j\,|j\rangle\otimes|i\rangle = |\phi\rangle\otimes|\psi\rangle$

$|\psi\rangle = \sum_i \psi_i|i\rangle$

$|\phi\rangle = \sum_j \phi_j|j\rangle$

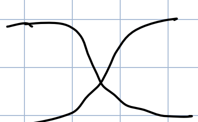
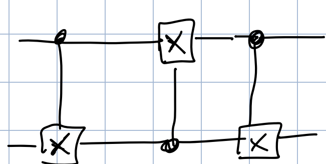
SWAP :=  = 

$\mathrm{SWAP}_{(A,B)} = \text{C-NOT}_{(A,B)}\ \ \text{C-NOT}_{(B,A)}\ \ \text{C-NOT}_{(A,B)}$

 = 

PROOF:

They act equally on $|00\rangle, |01\rangle, |10\rangle, |11\rangle$

 =  =