- $\Phi : L(\mathcal{H}_1) \longrightarrow L(\mathcal{H}_2)$ QUANTUM CHANNEL $\overset{\text{DEF}}{\Longleftrightarrow}$
  - LINEAR
    $\phi(\alpha \rho_1 + \beta \rho_2) = \alpha \phi(\rho_1) + \beta \phi(\rho_2)$
  - COMPLETELY POSITIVE
    - $A \in \mathcal{H}_1 \otimes \mathcal{H}_E$
    - $A \geq 0$

    $\Rightarrow (\Phi \otimes \mathbb{1})(A) \geq 0$
  - TRACE PRESERVING.
    $tr(\phi(X)) = tr(X)$

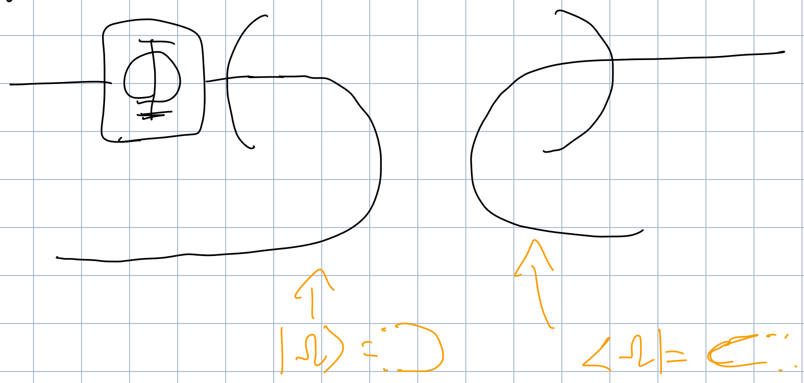
- DEF: CHOI - STATE of $\Phi(\cdot) : L(\mathcal{H}_A) \rightarrow L(\mathcal{H}_A)$

$$\rho_\Phi := (\Phi \otimes \underset{\uparrow}{\overset{H_B \text{ Space of same dimension } d_B}{\mathbb{1}}})(|\Omega\rangle\langle\Omega|) \quad \text{with} \quad |\Omega\rangle := \sum_{i=1}^{d} \frac{1}{d} \overset{\text{ortonormal basis states}}{|i\rangle \otimes |i\rangle}$$

We have $\rho_\Phi = \frac{1}{d}(\Phi \otimes \mathbb{1})\left(\sum_{i,j}^{d} |i\, i\rangle\langle j\, j|\right) = \frac{1}{d} \sum_{i,j=1}^{d} \Phi(|i\rangle\langle j|) \otimes |i\rangle\langle j|$

- Using diagrams a Choi state is:

$\rho_\Phi = \frac{1}{d} \cdot$



We have $\boxed{\rho_\Phi \geq 0 \iff \Phi(\cdot) \text{ is completely positive.}}$

PROOF:

"$\Leftarrow$" Easy: $\rho_\phi := \phi \otimes \mathbb{1}(|\Omega\rangle\langle\Omega|) \underset{\underset{\Phi \text{ compl. positive}}{\uparrow}}{\geq} 0$

"$\Rightarrow$": I have to show: $\rho_\phi \geq 0 \Rightarrow \left(\sigma \geq 0 \Rightarrow (\phi \otimes \mathbb{1})(\sigma) \geq 0\right)$

$$\Phi \otimes \underset{\underset{\dim(H_B) \geq \dim(H_A)}{\nearrow}}{\mathbb{1}} (\sigma) \underset{\underset{\sigma = \sum_{i=1}^{d_A} \underset{\underset{0}{\vee\!\!|}}{\lambda_i} |V_i\rangle\langle V_i|}{\uparrow}}{=} \sum_{i}^{d_A} \lambda_i (\phi \otimes \mathbb{1})(|V_i\rangle\langle V_i|)$$

Let's focus on $(\Phi \otimes \mathbb{1})(|V_i\rangle\langle V_i|)$.

$d_B \geq d_A \Rightarrow \min(d_A, d_B) = d_A$

- $$|V_i\rangle \underset{\uparrow}{=} \sum_{j=1}^{\min(d_A,d_B)} \underset{\overset{\vee}{0}}{s_j} |j\rangle \otimes |j\rangle \underset{\uparrow}{=} \sum_{j=1}^{\min(d_A,d_B)} X \otimes \mathbb{1} \; |j\rangle \otimes |j\rangle$$

  - $|V_i\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$
  - SCHMIDT DECOMP.

$(\exists \; X$ such that $X|j\rangle = s_j|j\rangle)$

$$\underset{\uparrow}{=} (X \otimes \mathbb{1}) \, W_A \otimes W_B |\Omega\rangle$$

- Where W_A and W_B are unitary such that: $\sum_j (W_A^\dagger |j\rangle) \otimes (W_B^\dagger |j\rangle) = |\Omega\rangle$

$$\underset{\uparrow}{=} X \otimes \mathbb{1} \left(\mathbb{1} \otimes W_B W_A^t\right) |\Omega\rangle =$$

TRANSPOSE-TRICK

$$\overset{\downarrow}{=} \mathbb{1} \otimes W_B W_A^t X^t |\Omega\rangle$$

- $$|V_i\rangle\langle V_i| = \left(\mathbb{1} \otimes W_B W_A^T X^T\right) |\Omega\rangle\langle\Omega| \left(\mathbb{1} \otimes \left(W_B W_A^t X^t\right)^\dagger\right)$$

- $$(\Phi \otimes \mathbb{1})\left[|V_i\rangle\langle V_i|\right] \overset{\downarrow}{=} \underbrace{\left(\mathbb{1} \otimes W_B W_A^T X^T\right)}_{\overset{\shortparallel}{M}} (\Phi \otimes \mathbb{1})(|\Omega\rangle\langle\Omega|) \left(\mathbb{1} \otimes \left(W_B W_A^t X^t\right)^\dagger\right)$$

$$= M \rho_\Phi M^\dagger$$

Now $\forall \, |\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ $$\langle\psi| M \rho_\Phi M^\dagger |\psi\rangle = \langle M^\dagger|\psi\rangle | \rho_\Phi | \underset{\overset{\shortparallel}{v}}{M^\dagger|\psi\rangle}\rangle$$

$$= \langle v | \rho_\Phi | v \rangle \underset{\uparrow}{\geq} 0$$

$\rho_\Phi \geq 0$

$\uparrow$

ASSUMPTION.

- Now we have proven that $\rho_\Phi \geq 0 \Rightarrow \Phi \otimes \underset{\uparrow}{\mathbb{1}} \geq 0$

H_B with $\dim(H_B) = \dim(H_A)$

- If $\dim(H_B) < \dim(H_A)$:

$$\phi \otimes \mathbb{1}_B(\sigma) = \operatorname{tr}_{B'}\left(\phi \otimes \underbrace{\mathbb{1}_B \otimes \mathbb{1}_{B'}}_{\text{Same dimension of A in total}}\left(\sigma \otimes |0\rangle_{B'}\langle 0|_{B'}\right)\right) \geq 0 \quad \Rightarrow \underline{\square}$$

- ρ_Φ uniquely define Φ using :

$$\Phi(\sigma) = d \operatorname{tr}_B\left(\mathbb{1} \otimes \sigma^t \; \rho_\Phi\right)$$

PROOF :

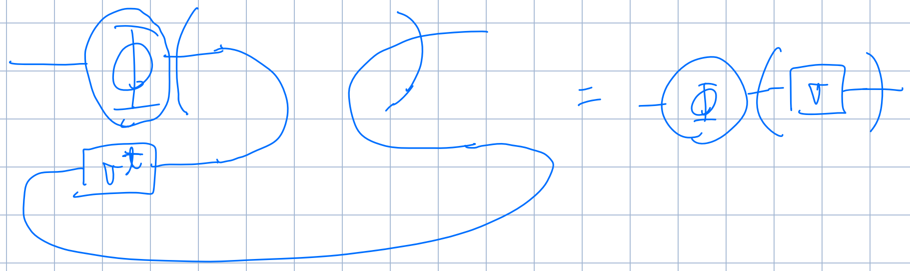
$$d \operatorname{tr}_B\left(\mathbb{1} \otimes \sigma^t \; \rho_\Phi\right) = \frac{d}{d} \operatorname{tr}_B\left(\mathbb{1} \otimes \sigma^t \sum_{i,j} \phi(|i\rangle\langle j|) \otimes |i\rangle\langle j|\right) =$$

$$= \sum_{i,j} \operatorname{tr}_B\left(\phi(|i\rangle\langle j|) \otimes \sigma^t \, |i\rangle\langle j|\right) =$$

$$= \sum_{i,j} \phi(|i\rangle\langle j|) \langle j|\sigma^t|i\rangle =$$

$$= \sum_{i,j} \phi(|i\rangle\langle j|) \langle i|\sigma|j\rangle \underset{\substack{\uparrow \\ \sum |i\rangle\langle i| = \mathbb{1}}}{=} \phi(\sigma)$$

Or Using diagrams:



- $\operatorname{tr}_A(\rho_\phi) = \frac{\mathbb{1}_B}{d}$ ⟸ Only the state ρ such that $\operatorname{tr}_A(\rho) = \frac{1}{d}$ can be Choi state.

PROOF :

$$\operatorname{tr}_A\left(\Phi \otimes \mathbb{1}\left(|\Omega\rangle\langle\Omega|\right)\right) = \frac{1}{d}\operatorname{tr}_A\left(\sum_{i,j}\phi(|i\rangle\langle j|) \otimes |i\rangle\langle j|\right) = \frac{1}{d}\sum_{i,j} \operatorname{tr}\left(\phi(|i\rangle\langle j|)\right) |i\rangle\langle j| =$$

$$\underset{\substack{\uparrow \\ \phi \text{ trace preserving}}}{=} \frac{1}{d}\sum_{i,j} \operatorname{tr}(|i\rangle\langle j|)\, |i\rangle\langle j| = \frac{1}{d}\sum_{i} |i\rangle\langle i| = \frac{\mathbb{1}}{d}$$

THEOREM KRAUS

- Φ quantum channel $\Longleftrightarrow$ $\exists \{K_i\}_{i=1}^{d^2} \in \mathcal{L}(H)$ such that :

$$\underset{\text{KRAUS OPERATORS}}{\{K_i\}} \qquad \begin{cases} \cdot\ \Phi(\rho) = \sum_{i=1}^{d^2} K_i \rho K_i^{+} \\ \cdot\ \sum_{i=1}^{d^2} K_i^{+} K_i = \mathbb{1} \end{cases}$$

PROOF:

" $\Longleftarrow$ "

- Linear : easy

- Trace preserving : $\mathrm{Tr}(\phi(\rho)) = \sum_{i=1}^{d^2} \mathrm{Tr}(K_i \rho K_i^{+}) = \sum_i \mathrm{Tr}(K_i^{+} K_i \rho) =$

$$= \mathrm{Tr}(\underbrace{\sum_i K_i^{+} K_i}_{\mathbb{1}} \rho) = \mathrm{Tr}(\rho)$$

- Completely positive: $\sigma \geq 0 \Rightarrow \forall |\psi\rangle \in \mathcal{H}_A \otimes H_B \quad \langle\psi| \Phi \otimes \mathbb{1}(\sigma)|\psi\rangle \geq 0$

SUBPROOF:

$$\langle\psi|(\Phi\otimes\mathbb{1})(\sigma)|\psi\rangle = \langle\psi|\sum_i K_i\otimes\mathbb{1}\ \sigma\ K_i^{+}\otimes\mathbb{1}|\psi\rangle =$$

$$= \sum_i \underbrace{\langle\psi|(K_i\otimes\mathbb{1})}_{\langle v_i|}\ \sigma\ \underbrace{(K_i^{+}\otimes\mathbb{1})|\psi\rangle}_{|v_i\rangle}$$

$$= \sum_i \langle v_i|\sigma|v_i\rangle \underset{\sigma\geq 0}{\overset{\uparrow}{\geq}} 0$$

" $\Longrightarrow$ "

- $\Phi(\rho) \underset{\rho=\rho^{+} \Rightarrow \text{ eigendec.}}{\overset{\uparrow}{=}} \sum_{i=1}^{d} \lambda_i \Phi(|v_i\rangle\langle v_i|)$

- $\Phi(|\psi\rangle\langle\psi|) = d\ {}_B\langle\psi^*|\ \rho_\Phi\ |\psi^*\rangle_B \equiv \textcircled{\star}$

SUB PROOF:

$${}_B\langle\psi^*|\ \rho_\Phi\ |\psi^*\rangle_B = {}_B\langle\psi^*|\ (\Phi\otimes\mathbb{1})(|\Omega\rangle\langle\Omega|)\ |\psi^*\rangle_B =$$

$$\overset{\uparrow}{=} \frac{1}{d}\sum_{i,j} \Phi(|i\rangle_A\langle j|)\ \langle\psi^*|i\rangle\ \langle j|\psi^*\rangle$$

$$\left(\Phi\otimes\mathbb{1}_B(|\Omega\rangle\langle\Omega|) = \frac{1}{d}\sum_{i,j}\Phi(|i\rangle_A{}_A\langle j|)\otimes|i\rangle_B{}_B\langle j| \right)$$

$$\underset{\uparrow}{=} \frac{1}{d} \sum_{i,j} \Phi\Big(\langle i|\psi\rangle |i\rangle\langle j| \langle\psi|j\rangle \Big) = \frac{1}{d}\Phi\big(|\psi\rangle\langle\psi|\big)$$

$$\left(\begin{aligned} \langle\psi^*|i\rangle &= \big(\langle i|\psi^*\rangle\big)^* \\ &= \langle i|^*|\psi\rangle \\ &\underset{\uparrow}{=} \langle i|\psi\rangle \end{aligned} \right)$$

$\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = |i\rangle$ comp. basis.

$\Rightarrow$ $\square$ SUBPROOF.

- $$\Phi(\rho) = \sum_{i=1}^{d} \lambda_i \, \Phi\big(|v_i\rangle\langle v_i|\big) = \sum_{i=1}^{d} \lambda_i \; {}_B\langle v_i^*|\, \rho_\Phi \,|v_i^*\rangle_B \, d$$

$$\underset{\uparrow}{=} \sum_{i=1}^{d} \lambda_i \sum_{j=1}^{d^2} p_j \; {}_B\langle v_i^*|\psi_j\rangle_{AB} \; {}_{AB}\langle\psi_j|v_i^*\rangle_B \, d =$$

$$\left(\rho_\Phi \geq 0 \Rightarrow \rho_\Phi^\dagger = \rho_\Phi \Rightarrow \text{diagonaliz.} \quad \Rightarrow \rho_\Phi = \sum_{j=1}^{d^2} p_j |\psi_j\rangle_{AB}\,{}_{AB}\langle\psi_j| \right)$$

$$= \sum_{j=1}^{d^2} p_j \, d \sum_{i=1}^{d} \lambda_i \; {}_B\langle v_i^*|\psi_j\rangle_{AB} \; {}_{AB}\langle\psi_j|v_i^*\rangle_B =$$

$$\underset{\uparrow}{=} \sum_{j=1}^{d^2} \sum_{i=1}^{d} \lambda_i \, K_j |v_i\rangle\langle v_i| K_j^\dagger =$$

$$\left(\underset{\uparrow}{K_j |\cdot\rangle_A} := {}_B\langle\cdot|^*|\psi_j\rangle_{AB} \sqrt{p_j}\sqrt{d} \quad \text{linear operators} \right)$$

$$\underset{\uparrow}{=} \sum_{j=1}^{d^2} K_j \,\rho\, K_j^\dagger$$

$\sum_{i=1}^{d} \lambda_i |v_i\rangle\langle v_i| = \rho$

- $$\mathrm{tr}\big(\Phi(\rho)\big) = 1 \Rightarrow \mathrm{tr}\Big(\sum_j K_j^\dagger K_j \rho\Big) = 1 \quad \forall \rho \Rightarrow \sum_{j=1}^{d^2} K_j^\dagger K_j = \mathbb{1}$$

STINESPRING REPRESENTATION.

$$\Phi(\cdot) \text{ quantum channel } \Longleftrightarrow \exists\; U_{AB} \in \mathcal{H}_A \otimes \overset{\substack{\text{auxiliary system}\\ \downarrow}}{\mathcal{H}_B} \text{ unitary such that}$$

$$\Phi(\rho) = \operatorname{tr}_B\left(U_{AB}\, \rho \otimes |0\rangle\langle 0|\, U_{AB}^{\dagger} \right)$$

PROOF:

"$\Longrightarrow$" Φ q. channel $\Longleftrightarrow$ $\exists \{K_i\}_{i=1}^{d^2}$ such that $\Phi(\rho) = \sum_{i=1}^{d^2} K_i \rho K_i^{\dagger}$ and $\sum_i K_i^{\dagger} K_i = \mathbb{1}$

CLAIM: $\underset{\nwarrow \text{ It's unitary}}{U_{AB}}\, |\phi\rangle \otimes |0\rangle = \sum_{i=1}^{d^2} K_i |\phi\rangle \otimes |i\rangle$

SUBPROOF:

$$U_{AB}\, \rho \otimes |0\rangle\langle 0|\, U_{AB}^{\dagger} \underset{\substack{\uparrow \\ \rho = \sum_i \lambda_i |v_i\rangle\langle v_i|}}{=} \sum_i \lambda_i\, U_{AB} \left(|v_i\rangle \otimes |0\rangle \right) \left(\langle v_i| \otimes \langle 0| \right) U_{AB}^{\dagger} =$$

$$= \sum_i \lambda_i \sum_{j=1}^{d^2} \left(K_j |v_i\rangle \otimes |j\rangle \right) \sum_{\ell=1}^{d^2} \left(\langle v_i| K_\ell^{\dagger} \otimes \langle \ell| \right)$$

$$\operatorname{tr}_B\left(U_{AB}\, \rho \otimes |0\rangle\langle 0|\, U_{AB}^{\dagger} \right) \overset{\downarrow}{=} \sum_i \lambda_i \sum_{j=1}^{d^2} \sum_{\ell=1}^{d^2} \left(K_j |v_i\rangle\langle v_i| K_\ell^{\dagger} \right) \underbrace{\operatorname{tr}_B\left(|j\rangle\langle \ell| \right)}_{\shortparallel\; \delta_{j,\ell}}$$

$$\underset{\substack{\uparrow \\ \sum \lambda_i |v_i\rangle\langle v_i| = \rho}}{=} \sum_{j=1}^{d^2} K_j \rho K_j^{\dagger}$$

"$\Longleftarrow$" $\Phi(\rho) = \operatorname{tr}_B\left(U_{AB}\, \rho \otimes |0\rangle\langle 0|\, U_{AB}^{\dagger} \right)$ is:

- linear : Yes!
- Trace preserving : Yes! ← $\operatorname{tr}\left(\Phi(\rho) \right) = \ldots = 1$
- Complet. positive : $\left(\Phi \otimes \underset{\mathcal{H}_2}{\mathbb{1}} \right) \left(\underset{\substack{\uparrow \\ \mathcal{H}_A \otimes \mathcal{H}_2}}{\sigma} \right) = \operatorname{tr}_B\left(U_{AB}\, \underset{\mathcal{H}_A \otimes \mathcal{H}_2}{\sigma} \otimes |0\rangle\langle 0|\, U_{AB}^{\dagger} \right) \geq 0.$