Quantum Field Theory I
Exercises

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1 Exercise 1: Gordon identities

Consider u and v, the positive and negative frequency solutions of the Dirac equation for a particle of mass m in momentum space, and two tetramomenta p and q for on-shell particles, i.e. $p^2 = q^2 = m^2$. Derive the Gordon identities for u [3pt] and v [3pt] :

$$\bar{u}(p)\gamma^\mu u(q) = \bar{u}(p)\left[\frac{(p+q)^\mu}{2m} + \frac{2i\sigma^{\mu\nu}(p-q)_\nu}{2m}\right]u(q) \tag{1.1}$$

$$\bar{v}(p)\gamma^\mu v(q) = -\bar{v}(p)\left[\frac{(p+q)^\mu}{2m} + \frac{2i\sigma^{\mu\nu}(p-q)_\nu}{2m}\right]v(q) \tag{1.2}$$

where γ^μ are matrices satisfying the Clifford algebra and $\sigma^{\mu\nu} = i\left[\gamma^\mu, \gamma^\nu\right]/4$

Solution

Gordon equation for u We have:

$$2i\sigma^{\mu\nu} = \eta^{\mu\nu}\mathbb{1} - \gamma^\mu\gamma^\nu \tag{1.3}$$

Proof.

$$\sigma^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu,\gamma^\nu\right] = \frac{i}{4}\left(2\gamma^\mu\gamma^\nu - \{\gamma^\mu,\gamma^\nu\}\right) = \frac{i}{4}\left(2\gamma^\mu\gamma^\nu - 2\eta^{\mu\nu}\mathbb{1}\right) = \frac{i}{2}\left(\gamma^\mu\gamma^\nu - \eta^{\mu\nu}\mathbb{1}\right)$$

where we have used $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}$. ■

Putting this result into RHS of eq(1.1):

$$\begin{gathered}\frac{\bar{u}(p)}{2m}\left[(p+q)^\mu + 2i\sigma^{\mu\nu}(p-q)_\nu\right]u(q) = \frac{\bar{u}(p)}{2m}\left[(p+q)^\mu + \eta^{\mu\nu}(p-q)_\nu - \gamma^\mu\gamma^\nu(p-q)_\nu\right]u(q) = \\ \frac{\bar{u}(p)}{2m}\left[(p+q)^\mu + (p-q)^\mu - \gamma^\mu\gamma^\nu p_\nu + \gamma^\mu\gamma^\nu q_\nu\right]u(q) = \frac{\bar{u}(p)}{2m}\left[2p^\mu - \gamma^\mu\gamma^\nu p_\nu + \gamma^\mu\gamma^\nu q_\nu\right]u(q)\end{gathered} \tag{1.4}$$

Now, we have:

$$\gamma^\mu\gamma^\nu p_\nu = 2p^\mu - \gamma^\nu p_\nu\gamma^\mu \tag{1.5}$$

Proof.

$$\gamma^\mu\gamma^\nu p_\nu = \{\gamma^\mu,\gamma^\nu\}p_\nu - \gamma^\nu\gamma^\mu p_\nu = 2\eta^{\mu\nu}p_\nu - \gamma^\nu p_\nu\gamma^\mu = 2p^\mu - \gamma^\nu p_\nu\gamma^\mu$$

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Hence:

$$\begin{gathered}\frac{\bar{u}(p)}{2m}\left[(p+q)^\mu + 2i\sigma^{\mu\nu}(p-q)_\nu\right]u(q) = \frac{\bar{u}(p)}{2m}\left[2p^\mu - 2p^\mu + \gamma^\nu p_\nu\gamma^\mu + \gamma^\mu\gamma^\nu q_\nu\right]u(q) = \\ \frac{\bar{u}(p)}{2m}\left[\gamma^\nu p_\nu\gamma^\mu + \gamma^\mu\gamma^\nu q_\nu\right]u(q)\end{gathered} \tag{1.6}$$

Using the Dirac equation in momentum space and the one for the Dirac adjoint spinor (for positive frequencies):

$$\begin{aligned} \gamma^\nu q_\nu u(q) &= m u(q) \\ \bar{u}(p)\gamma^\nu p_\nu &= m\bar{u}(p) \end{aligned} \tag{1.7}$$

we get:

$$\frac{\bar{u}(p)}{2m}\left[\gamma^\nu p_\nu \gamma^\mu + \gamma^\mu \gamma^\nu q_\nu\right] u(q) = \frac{\bar{u}(p)}{2m}\left[m\gamma^\mu + \gamma^\mu m\right] u(q) = \bar{u}(p)\gamma^\mu u(q) \tag{1.8}$$

And so:

$$\bar{u}(p)\gamma^\mu u(q) = \bar{u}(p)\left[\frac{(p+q)^\mu}{2m} + \frac{2i\sigma^{\mu\nu}(p-q)_\nu}{2m}\right] u(q) \tag{1.9}$$

Gordon equation for v Since we did not use u in the calculation before eq.(1.6), we can recycle all what we wrote for u utill this point for the current case .

Hence:

$$\frac{\bar{v}(p)}{2m}\left[(p+q)^\mu + 2i\sigma^{\mu\nu}(p-q)_\nu\right] v(q) = \frac{\bar{v}(p)}{2m}\left[\gamma^\nu p_\nu \gamma^\mu + \gamma^\mu \gamma^\nu q_\nu\right] v(q) \tag{1.10}$$

Using the Dirac equation in momentum space and the one for the Dirac adjoint spinor (for negative frequencies):

$$\begin{aligned} \gamma^\nu q_\nu v(q) &= -m v(q) \\ \bar{v}(p)\gamma^\nu p_\nu &= -m\bar{v}(p) \end{aligned} \tag{1.11}$$

we get:

$$\frac{\bar{v}(p)}{2m}\left[\gamma^\nu p_\nu \gamma^\mu + \gamma^\mu \gamma^\nu q_\nu\right] v(q) = \frac{\bar{v}(p)}{2m}\left[-m\gamma^\mu - \gamma^\mu m\right] v(q) = -\bar{v}(p)\gamma^\mu v(q) \tag{1.12}$$

And so:

$$\bar{v}(p)\gamma^\mu v(q) = -\bar{v}(p)\left[\frac{(p+q)^\mu}{2m} + \frac{2i\sigma^{\mu\nu}(p-q)_\nu}{2m}\right] v(q) \tag{1.13}$$

2 Exercise 2: free vectorial massive field

Starting from a real vector field A^μ, the Proca density of Lagrangian is obtained by adding to the Maxwell Lagrangian a mass term:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu \tag{2.1}$$

with $m > 0$. Discuss the quantization of such a theory. This exercise can be solved in different ways and the student can choose one of them (e.g. the canonical quantization, as done in the lectures for the electromagnetic field, or the particle quantization, as done for the scalar field). It is important to discuss:

- the gauge invariance of the Lagrangian: is this Lagrangian invariant under $A_\mu \to A_\mu + \partial_\mu f$? [1pt];
- the equations of motion and their solutions [4pt]; hint: compute the Euler- Lagrange equations from the density of Lagrangian. Apply ∂_ν to the equation for A^ν to get an additional condition for $\partial_\nu A^\nu$;
- the physical degrees of freedom of the quantized particles [3pt];
- the commutation (or anticommutation) relation in spacetime and the Feynman propagator [4pt].

General hint: A closer analogy to the massless case can be misleading. Try not to modify the Lagrangian and not to use an approach similar to the Gupta- Bleuler one. In particular, try to quantize only the relevant degrees of freedom.

Solution

The transformation

$$A_\mu \to A_\mu + \partial_\mu f \tag{2.2}$$

is not a symmetry of the the Lagrangian (2.1), in fact:

$$\mathcal{L} \to -\frac{1}{4}F'^{\mu\nu}F'_{\mu\nu} + \frac{1}{2}m^2 A'_\mu A'^{\,\mu} \tag{2.3}$$

we have:

$$-\frac{1}{4}F'^{\mu\nu}F'_{\mu\nu} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{2.4}$$

Proof. Since:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.5}$$

we have:

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = \partial_\mu A_\nu + \partial_\mu\partial_\nu f - \partial_\nu A_\mu - \partial_\nu\partial_\mu f = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu} \tag{2.6}$$

Therefore:

$$F'^{\mu\nu}F'_{\mu\nu} = F^{\mu\nu}F_{\mu\nu} \tag{2.7}$$

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while:

$$\frac{1}{2}m^2 A'_\mu A'^\mu = \frac{1}{2}m^2 A_\mu A^\mu + m^2 \partial_\mu f A^\mu + m^2 \partial^\mu f \partial_\mu f \tag{2.8}$$

So in general:

$$S' = \int \mathcal{L}' \, d^4x \neq \int \mathcal{L} \, d^4x = S \tag{2.9}$$

Writing explicitly:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.10}$$

the density of Lagrangian 2.1 could be put in the form :

$$\mathcal{L} = -\frac{1}{2}\partial^\mu A^\nu \partial_\mu A_\nu + \frac{1}{2}\partial^\mu A^\nu \partial_\nu A_\mu + \frac{1}{2}m^2 A_\mu A^\mu \tag{2.11}$$

The Euler-Lagrange equations are:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu A_\nu\right)} \right) = \frac{\partial \mathcal{L}}{\partial A_\nu} \tag{2.12}$$

The equations of motion are :

$$-\Box A^\nu + \partial^\nu \partial_\mu A^\mu = m^2 A^\nu \tag{2.13}$$

Proof. Computing LHS of eq(2.12):

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu A_\nu\right)} &= -\frac{1}{2}\frac{\partial \left(\partial^\alpha A^\beta \partial_\alpha A_\beta\right)}{\partial \left(\partial_\mu A_\nu\right)} + \frac{1}{2}\frac{\partial \left(\partial^\alpha A^\beta \partial_\beta A_\alpha\right)}{\partial \left(\partial_\mu A_\nu\right)} + \frac{1}{2}m^2 \frac{\partial \left(A_\alpha A^\alpha\right)}{\partial \left(\partial_\mu A_\nu\right)} \\
&= -\frac{1}{2}\eta^{\alpha l}\eta^{\beta m}\frac{\partial \left(\partial_l A_m \partial_\alpha A_\beta\right)}{\partial \left(\partial_\mu A_\nu\right)} + \frac{1}{2}\eta^{\alpha l}\eta^{\beta m}\frac{\partial \left(\partial_l A_m \partial_\beta A_\alpha\right)}{\partial \left(\partial_\mu A_\nu\right)} \\
&= -\frac{1}{2}\eta^{\alpha l}\eta^{\beta m}\left(\delta_{l\mu}\delta_{m\nu}\partial_\alpha A_\beta + \partial_l A_m \delta_{\alpha\mu}\delta_{\beta\nu}\right) + \frac{1}{2}\eta^{\alpha l}\eta^{\beta m}\left(\delta_{l\mu}\delta_{m\nu}\partial_\beta A_\alpha + \partial_l A_m \delta_{\beta\mu}\delta_{\alpha\nu}\right) \\
&= -\partial^\mu A^\nu + \partial^\nu A^\mu
\end{aligned} \tag{2.14}$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu A_\nu\right)} \right) = -\partial_\mu \partial^\mu A^\nu + \partial_\mu \partial^\nu A^\mu = -\Box A^\nu + \partial^\nu \partial_\mu A^\mu \tag{2.15}$$

Computing RHS:

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = \frac{1}{2}m^2 \frac{\partial \left(A_\alpha A^\alpha\right)}{\partial A_\nu} = \frac{1}{2}m^2 \eta^{\alpha l}\frac{\partial \left(A_\alpha A_l\right)}{\partial A_\nu} = m^2 A^\nu \tag{2.16}$$

Therefore, the equations of motion are :

$$-\Box A^\nu + \partial^\nu \partial_\mu A^\mu = m^2 A^\nu \tag{2.17}$$

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We have:

$$\partial_\nu A^\nu = 0 \tag{2.18}$$

Proof. Applying ∂_ν to both side of eq.(2.13) and commuting the derivatives:

$$m^2 \partial_\nu A^\nu = -\Box \partial_\nu A^\nu + \partial_\nu \partial^\nu \partial_\mu A^\mu = -\Box \partial_\nu A^\nu + \Box \partial_\nu A^\nu = 0 \tag{2.19}$$

So if $m \neq 0$, we get the thesis.

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Considering eq.(2.18), equations of motion became the well known Klein-Gordon ones:

$$\left(\Box + m^2\right) A^\nu = 0 \qquad \partial_\nu A^\nu = 0 \tag{2.20}$$

where each components A^ν satisfies K.G. equation for mass scalar field which solutions are:

$$A^\nu(x) = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2E_{\mathbf{k}}}} \left(c^\nu(\vec{k})e^{-ikx} + c^{*\nu}(\vec{k})e^{ikx}\right) \tag{2.21}$$

with:

$$k_0^2 - |\vec{k}|^2 = m^2 \tag{2.22}$$

I define (I am simply doing a basis change):

$$c^\nu(\vec{k}) = \sum_{r=0}^{3} a(\vec{k},r)\epsilon_r^\nu(\vec{k}) \tag{2.23}$$

where $\epsilon_r^\nu(\vec{k})$ is a basis of our Minkowski space such that we start to choose as:

$$\epsilon_0^\mu(\vec{k}) = (1,0,0,0) \tag{2.24}$$

$$\epsilon_3^\mu(\vec{k}) = (0, \frac{\vec{k}}{|\vec{k}|}) \tag{2.25}$$

$$\epsilon_{1,2}^\mu(\vec{k}) = (0, \vec{\epsilon}_{1,2}) \text{ with } \vec{\epsilon}_i \cdot \vec{\epsilon}_j = \delta_{ij} \text{ and } \vec{\epsilon}_i \wedge \vec{\epsilon}_j = \epsilon^{ijk}\vec{\epsilon}_k \text{ where } i = j = 1,2,3 \tag{2.26}$$

We have:

$$\epsilon_r^\mu(\vec{k})\epsilon_{\mu\,s}^*(\vec{k}) = -\alpha_r\delta_{r,s} \text{ with } \begin{cases} \alpha_r = -1, \text{if } r = 0 \\ \alpha_r = +1, \text{if } r = 1,2,3 \end{cases} \tag{2.27}$$

Completeness relation is:

$$\sum_{r=0}^{3} \alpha_r \epsilon_r^\mu(\vec{k})\epsilon_r^{\nu *}(\vec{k}) = -\eta^{\mu\nu} \tag{2.28}$$

Therefore:

$$A^\nu(x) = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2E_{\mathbf{k}}}} \sum_{r=0}^{3} \left(a(\vec{k},r)\epsilon_r^\nu(\vec{k})e^{-ikx} + a^*(\vec{k},r)\epsilon_r^{*\nu}(\vec{k})e^{ikx}\right) \tag{2.29}$$

But we have not yet imposed the condition 2.18 in our general solution. Let's do it:

$$0 = \partial_\nu A^\nu = \int \frac{-i\,d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2E_{\mathbf{k}}}} \sum_{r=0}^{3} \left(a(\vec{k},r)\epsilon_r^\nu(\vec{k})k_\nu e^{-ikx} - a^*(\vec{k},r)\epsilon_r^{*\nu}(\vec{k})k_\nu e^{ikx}\right) \tag{2.30}$$

hence, in order to satisfy the last equation, we must have:

$$\sum_{r=0}^{3} a(\vec{k},r)\epsilon_r^\nu(\vec{k})k_\nu = 0 \tag{2.31}$$

$$0 = a(\vec{k},0)\epsilon_0^\nu(\vec{k})k_\nu + a(\vec{k},1)\epsilon_1^\nu(\vec{k})k_\nu + a(\vec{k},2)\epsilon_2^\nu(\vec{k})k_\nu + a(\vec{k},3)\epsilon_3^\nu(\vec{k})k_\nu \tag{2.32}$$

$$= a(\vec{k},0)k^0 - a(\vec{k},1)\vec{\epsilon}_1 \cdot \vec{k} - a(\vec{k},2)\vec{\epsilon}_2 \cdot \vec{k} - a(\vec{k},3)|\vec{k}| \tag{2.33}$$

$$= a(\vec{k},0)k^0 + 0 + 0 - a(\vec{k},3)|\vec{k}| \tag{2.34}$$

So, we must have:

$$a(\vec{k},0) = a(\vec{k},3)\frac{|\vec{k}|}{k^0} = a(\vec{k},3)\frac{|\vec{k}|}{\sqrt{|\vec{k}|^2+m^2}} \tag{2.35}$$

Thus we have:

$$\sum_{r=0}^{3} a(\vec{k},r)\epsilon_r^\nu(\vec{k}) = \sum_{r=1}^{2} a(\vec{k},r)\epsilon_r^\nu(\vec{k}) + a(\vec{k},0)\epsilon_0^\nu(\vec{k}) + a(\vec{k},3)\epsilon_3^\nu(\vec{k}) \tag{2.36}$$

$$= \sum_{r=1}^{2} a(\vec{k},r)\epsilon_r^\nu(\vec{k}) + a(\vec{k},3)\frac{|\vec{k}|}{k^0}\epsilon_0^\nu(\vec{k}) + a(\vec{k},3)\epsilon_3^\nu(\vec{k}) \tag{2.37}$$

$$= \sum_{r=1}^{2} a(\vec{k},r)\epsilon_r^\nu(\vec{k}) + a(\vec{k},3)\left(\frac{|\vec{k}|}{k^0}\epsilon_0^\nu(\vec{k}) + \epsilon_3^\nu(\vec{k})\right) \tag{2.38}$$

$$= \sum_{r=1}^{2} a(\vec{k},r)\epsilon_r^\nu(\vec{k}) + a(\vec{k},3)\left(\frac{|\vec{k}|}{k^0}(1,\vec{0}) + (0,\frac{\vec{k}}{|\vec{k}|})\right) \tag{2.39}$$

$$= \sum_{r=1}^{2} a(\vec{k},r)\epsilon_r^\nu(\vec{k}) + a(\vec{k},3)\left((\frac{|\vec{k}|}{k^0},\frac{\vec{k}}{|\vec{k}|})\right) \tag{2.40}$$

So, defining:

$$\tilde{\epsilon}_1^\nu(\vec{k}) = \epsilon_1^\mu(\vec{k}) \tag{2.41}$$

$$\tilde{\epsilon}_2^\nu(\vec{k}) = \epsilon_2^\mu(\vec{k}) \tag{2.42}$$

$$\tilde{\epsilon}_3^\nu(\vec{k}) = (\frac{|\vec{k}|}{k^0},\frac{\vec{k}}{|\vec{k}|}) \tag{2.43}$$

Hence:

$$\sum_{r=0}^{3} a(\vec{k},r)\epsilon_r^\nu(\vec{k}) = \sum_{\lambda=1}^{3} a(\vec{k},\lambda)\tilde{\epsilon}_\lambda^\nu(\vec{k}) \tag{2.44}$$

$$\sum_{r=0}^{3} a^*(\vec{k},r)\epsilon_r^{*\nu}(\vec{k}) = \sum_{\lambda=1}^{3} a^*(\vec{k},\lambda)\tilde{\epsilon}_\lambda^{*\nu}(\vec{k}) \tag{2.45}$$

Consequently, we can rewrite eq(2.29) as:

$$\begin{aligned}
A^\nu(x) &= \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2E_{\mathbf{k}}}} \sum_{r=0}^{3}\left(a(\vec{k},r)\epsilon_r^\nu(\vec{k})e^{-ikx} + a^*(\vec{k},r)\epsilon_r^{*\nu}(\vec{k})e^{ikx}\right) = \\
&= \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2E_{\mathbf{k}}}} \left(\left(\sum_{r=0}^{3} a(\vec{k},r)\epsilon_r^\nu(\vec{k})\right)e^{-ikx} + \left(\sum_{r=0}^{3} a^*(\vec{k},r)\epsilon_r^{*\nu}(\vec{k})\right)e^{ikx}\right) = \\
&= \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2E_{\mathbf{k}}}} \left(\left(\sum_{\lambda=1}^{3} a(\vec{k},\lambda)\tilde{\epsilon}_\lambda^\nu(\vec{k})\right)e^{-ikx} + \left(\sum_{\lambda=1}^{3} a^*(\vec{k},\lambda)\tilde{\epsilon}_\lambda^{*\nu}(\vec{k})\right)e^{ikx}\right) = \\
&= \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2E_{\mathbf{k}}}} \sum_{\lambda=1}^{3}\left(a(\vec{k},\lambda)\tilde{\epsilon}_\lambda^\nu(\vec{k})e^{-ikx} + a^*(\vec{k},\lambda)\tilde{\epsilon}_\lambda^{*\nu}(\vec{k})e^{ikx}\right)
\end{aligned} \tag{2.46}$$

In order to have all $\tilde{\epsilon}^{\nu}_{\lambda}(\vec{k})$ that satisfy $\tilde{\epsilon}^{\nu}_{\lambda}(\vec{k})\tilde{\epsilon}^{*}_{\nu\lambda}(\vec{k}) = -1$ we scale the basis element:

$$\tilde{\epsilon}^{\nu}_{3}(\vec{k}) = (\frac{|\vec{k}|}{k^0}, \frac{\vec{k}}{|\vec{k}|}) \rightarrow \tilde{\epsilon}^{\nu}_{3}(\vec{k}) = \frac{k^0}{m}(\frac{|\vec{k}|}{k^0}, \frac{\vec{k}}{|\vec{k}|}) \tag{2.47}$$

In this way (using "on shell" relation eq(2.22)):

$$\tilde{\epsilon}^{\nu}_{3}(\vec{k})\tilde{\epsilon}_{\nu 3}(\vec{k}) = \left(\frac{k^0}{m}\right)^2 \left(\left(\frac{|\vec{k}|}{k^0}\right)^2 - 1\right) = \frac{|\vec{k}|^2 - k^{0\,2}}{m^2} = -1 \tag{2.48}$$

For a simpler notiation, we define (this is a notation abuse since we defined in a very similar way vectors in eq (2.26)):

$$\epsilon^{\nu}(\vec{k}, \lambda) = \tilde{\epsilon}^{\nu}_{\lambda}(\vec{k}) \tag{2.49}$$

To sum up, our new 3 dimensional basis of vectors are:

$$\epsilon^{\nu}(\vec{k}, 3) = \frac{k^0}{m}(\frac{|\vec{k}|}{k^0}, \frac{\vec{k}}{|\vec{k}|}) \tag{2.50}$$

$$\epsilon^{\nu}(\vec{k}, \lambda = 1, 2) = (0, \vec{\epsilon}_{1,2}) \text{ with } \vec{\epsilon}_i \cdot \vec{\epsilon}_j = \delta_{ij} \text{ and } \vec{\epsilon}_i \wedge \vec{\epsilon}_j = \epsilon^{ijk}\vec{\epsilon}_k \tag{2.51}$$

Useful properties are:

$$\epsilon^{\nu}(\vec{k}, \lambda)\epsilon_{\nu}(\vec{k}, \lambda) = -1 \tag{2.52}$$

$$\epsilon^{\nu}(\vec{k}, \lambda)k_{\nu} = 0 \tag{2.53}$$

From these equations we can get the completeness relation:

$$\sum_{\lambda=1}^{3} \epsilon^{\mu}(\vec{k}, \lambda)\epsilon^{\nu *}(\vec{k}, \lambda) = -\eta^{\mu\nu} + \frac{k^{\mu}k^{\nu}}{m^2} \tag{2.54}$$

Proof. The set made by vectors which appear in eq(2.51) and $\frac{k^{\nu}}{m} \equiv \epsilon^{\mu}(\vec{k}, S)$ are 4 orthogonal ones (for eq(2.53)). Moreover $\epsilon^{\mu}(\vec{k}, S)\epsilon^{*}_{\mu}(\vec{k}, S) = 1$ for on shell condition. So $\epsilon^{\mu}(\vec{k}, \lambda)$ for $\lambda = 1, 2, 3, S$ form an orthonormal basis for four-vectors in Minkowski space. So they must satisfy the completeness relation of our Minkowsky space:

$$\eta^{\mu\nu} = \epsilon^{\mu}(\vec{k}, S)\epsilon^{\nu *}(\vec{k}, S) - \sum_{\lambda=1}^{3} \epsilon^{\mu}(\vec{k}, \lambda)\epsilon^{\nu *}(\vec{k}, \lambda) = \frac{k^{\mu}}{m}\frac{k^{\nu}}{m} - \sum_{\lambda=1}^{3} \epsilon^{\mu}(\vec{k}, \lambda)\epsilon^{\nu *}(\vec{k}, \lambda) \tag{2.55}$$

■

So, we have:

$$A^{\nu}(x) = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2E_{\mathbf{k}}}} \sum_{\lambda=1}^{3} \left(a(\vec{k}, \lambda)\epsilon^{\nu}(\vec{k}, \lambda)e^{-ikx} + a^{*}(\vec{k}, \lambda)\epsilon^{\nu *}(\vec{k}, \lambda)e^{ikx}\right) \tag{2.56}$$

Therefore, we have just shown that tetravectors we need to describe our fields A^{ν} are three instead of four. We have momentum:

$$\Pi^{\nu}_{A} = \frac{\partial \mathcal{L}}{\partial \dot{A}_{\nu}} = \begin{cases} 0 & \text{for } \nu = 0 \\ \partial^{\nu}A^{0} - \partial^{0}A^{\nu} = -F^{0\nu} \equiv E^{\nu} & \text{for } \nu = 1, 2, 3 \end{cases} \tag{2.57}$$

Proof. It can be shown easily doing an explicit calculation. ■

Now that we have developed our classical theory, let's quantize it.

Quantization To quantize the theory, we are going to use the particle approach. Firstly, it must be pointed out that the constraint 2.18 tells that of the four A^μ only three of them are independent, and covariantly describe the three polarizations associated with the particle of spin 1 and mass $m \neq 0$. So the physical degrees of freedom to quantize are 3. Hence, we want to quantize a theory which Poincare group rappresentation on single particle state is labeled by $(m \neq 0, S = 1)$.

From spin statistic theorem, we know that these particles obey to Bose statistics, which means that states of particles are completely simmetric respect to the exchange of them.

Let's start introduce the Fock space of our theory as the direct sum of Hilbert spaces which describe every number of particles space:

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathscr{H}_n \tag{2.58}$$

$\mathscr{H}_0$ is the Hilbert space which describe the one composed by 0 particles:

$$Basis\,(\mathscr{H}_0) = \{|0\rangle\} \tag{2.59}$$

where $|0\rangle$ is the so called "vacuum state" that satisfies $\langle 0|0\rangle = 1$ and we assume that under Lorentz-Poincarè transformation it is invariant, which means $\hat{U}(\Lambda, a)\,|0\rangle = |0\rangle$.

$$Basis\,(\mathscr{H}_1) = \{|\vec{p}, \lambda\rangle\} \quad \text{with: } \vec{p} \in \mathbb{R}^3, E_p \equiv p^0 = \sqrt{|\vec{p}|^2 + m^2}, \lambda = 1, 2, 3 \tag{2.60}$$

We assume the following normalization:

$$\langle \vec{p}, r|\vec{q}, s\rangle = 2E_p \delta_{rs} \delta(\vec{p} - \vec{q}) \tag{2.61}$$

Since set 2.60 is a basis of our Hilbert space the following completeness relation must be valid:

$$\mathbb{1}_{\mathscr{H}_1} = \sum_{\lambda=1}^{3} \int \frac{d^3p}{2E_p} \,|\vec{p}, \lambda\rangle \langle \vec{p}, \lambda| \tag{2.62}$$

We postulate the excistance of a family of non-hermitian operators

$$\{\hat{a}(\vec{p}, \lambda)\}_{\substack{\vec{p} \in \mathbb{R}^3 \\ \lambda = 1,2,3}} \tag{2.63}$$

such that:

$$|\vec{p}, \lambda\rangle = \sqrt{2E_p}\hat{a}^\dagger(\vec{p}, \lambda)\,|0\rangle \tag{2.64}$$

$$\hat{a}(\vec{p}, \lambda)\,|0\rangle = 0 \tag{2.65}$$

and for the previous properties we call $\hat{a}(\vec{p}, \lambda)$ annihilation operators while $\hat{a}^\dagger(\vec{p}, \lambda)$ creation operators.

$$\hat{a}(\vec{p}, r)\,|\vec{q}, s\rangle = \sqrt{2E_p}\delta_{rs}\delta\,(\vec{p} - \vec{q})\,|0\rangle \tag{2.66}$$

Proof.

$$\begin{aligned}\langle 0|\sqrt{2E_p}\hat{a}(\vec{p}, r)\,|\vec{q}, s\rangle &= (\sqrt{2E_p}\hat{a}^\dagger(\vec{p}, r)\,|0\rangle)^\dagger\,|\vec{q}, s\rangle = \langle \vec{p}, r|\vec{q}, s\rangle = \\ &= 2E_p\delta_{rs}\delta(\vec{p} - \vec{q})\,\langle 0|0\rangle = \langle 0|\,2E_p\delta_{rs}\delta(\vec{p} - \vec{q})\,|0\rangle\end{aligned}$$

Hence:

$$\sqrt{2E_p}\hat{a}(\vec{p}, r)\,|\vec{q}, s\rangle = 2E_p\delta_{rs}\delta(\vec{p} - \vec{q})\,|0\rangle$$

■

$$Basis\,(\mathscr{H}_2) = \{|\vec{p_1}, \lambda_1; \vec{p_2}, \lambda_2\rangle\} \quad \text{with: } \vec{p_i} \in \mathbb{R}^3, E_{p_i} \equiv p_i^0 = \sqrt{|\vec{p_i}|^2 + m^2}, \tag{2.67}$$

$$\lambda_i = 1, 2, 3;\ \ i = 1, 2 \tag{2.68}$$

Considering that we have bosonic particles, it must be true that:

$$|\vec{p_1}, \lambda_1; \vec{p_2}, \lambda_2\rangle = |\vec{p_2}, \lambda_2; \vec{p_1}, \lambda_1\rangle \tag{2.69}$$

We assume the following normalization (since we have bosonic particles there must be the "+" between the two terms down here):

$$\langle \vec{p_1}, r_1; \vec{p_2}, r_2 | \vec{q_1}, s_1; \vec{q_2}, s_2\rangle = \tag{2.70}$$

$$= 2E_{p_1} 2E_{p_2} \left(\delta_{r_1 s_1}\delta_{r_2 s_2}\delta\left(\vec{p_1} - \vec{q_1}\right)\delta\left(\vec{p_2} - \vec{q_2}\right) + \delta_{r_1 s_2}\delta_{r_2 s_1}\delta\left(\vec{p_1} - \vec{q_2}\right)\delta\left(\vec{p_2} - \vec{q_1}\right)\right) \tag{2.71}$$

Given the fact that set 2.68 is a basis of our Hilbert space the following completeness relation must be valid:

$$\mathbb{1}_{\mathscr{H}_2} = \sum_{\lambda_1=1}^{3}\sum_{\lambda_2=1}^{3}\int \frac{d^3p_1}{2E_{p_1}}\frac{d^3p_2}{2E_{p_2}}\, |\vec{p_1}, \lambda_1; \vec{p_2}, \lambda_2\rangle \langle \vec{p_1}, \lambda_1; \vec{p_2}, \lambda_2| \tag{2.72}$$

We can state :

$$|\vec{p_1}, \lambda_1; \vec{p_2}, \lambda_2\rangle = \sqrt{2E_{p_1}}\hat{a}^\dagger(\vec{p_1}, \lambda_1)\,|\vec{p_2}, \lambda_2\rangle = \sqrt{2E_{p_1}}\sqrt{2E_{p_2}}\hat{a}^\dagger(\vec{p_1}, \lambda_1)\hat{a}^\dagger(\vec{p_2}, \lambda_2)\,|0\rangle \tag{2.73}$$

$$\hat{a}(\vec{p}, \lambda)\,|\vec{p_1}, \lambda_1; \vec{p_2}, \lambda_2\rangle = \sqrt{2E_p}\left(\delta_{\lambda\lambda_1}\delta\left(\vec{p} - \vec{p_1}\right)|\vec{p_2}, \lambda_2\rangle + \delta_{\lambda\lambda_2}\delta\left(\vec{p} - \vec{p_2}\right)|\vec{p_1}, \lambda_1\rangle\right) \tag{2.74}$$

Proof.

$$\begin{aligned}
&\left\langle \vec{p'}, \lambda' \right| \sqrt{2E_p}\hat{a}(\vec{p}, \lambda)\,|\vec{p_1}, \lambda_1; \vec{p_2}, \lambda_2\rangle = \left\langle \vec{p}, \lambda; \vec{p'}, \lambda' \middle| \vec{p_1}, \lambda_1; \vec{p_2}, \lambda_2 \right\rangle \\
&= 2E_p 2E_{p'}\left(\delta_{\lambda\lambda_1}\delta_{\lambda'\lambda_2}\delta\left(\vec{p} - \vec{p_1}\right)\delta\left(\vec{p'} - \vec{p_2}\right) + \delta_{\lambda\lambda_2}\delta_{\lambda'\lambda_1}\delta\left(\vec{p} - \vec{p_2}\right)\delta\left(\vec{p'} - \vec{p_1}\right)\right) = \\
&= 2E_p\left(\delta_{\lambda\lambda_1}\delta\left(\vec{p} - \vec{p_1}\right)\left\langle \vec{p'}, \lambda' \middle| \vec{p_2}, \lambda_2\right\rangle + \delta_{\lambda\lambda_2}\delta\left(\vec{p} - \vec{p_2}\right)\left\langle \vec{p'}, \lambda' \middle| \vec{p_1}, \lambda_1\right\rangle\right) = \\
&= \left\langle \vec{p'}, \lambda' \right| 2E_p\left(\delta_{\lambda\lambda_1}\delta\left(\vec{p} - \vec{p_1}\right)|\vec{p_2}, \lambda_2\rangle + \delta_{\lambda\lambda_2}\delta\left(\vec{p} - \vec{p_2}\right)|\vec{p_1}, \lambda_1\rangle\right)
\end{aligned}$$

Hence:

$$\sqrt{2E_p}\hat{a}(\vec{p}, \lambda)\,|\vec{p_1}, \lambda_1; \vec{p_2}, \lambda_2\rangle = 2E_p\left(\delta_{\lambda\lambda_1}\delta\left(\vec{p} - \vec{p_1}\right)|\vec{p_2}, \lambda_2\rangle + \delta_{\lambda\lambda_2}\delta\left(\vec{p} - \vec{p_2}\right)|\vec{p_1}, \lambda_1\rangle\right)$$

■

Now, we are going to show the commutation relations of our creation and annihilation operators:

$$[\hat{a}(\vec{p_1}, \lambda_1), \hat{a}(\vec{p_2}, \lambda_2)] = \left[\hat{a}^\dagger(\vec{p_1}, \lambda_1), \hat{a}^\dagger(\vec{p_2}, \lambda_2)\right] = 0 \tag{2.75}$$

Proof.

$$\sqrt{2E_{p_1}}\sqrt{2E_{p_2}}\hat{a}^\dagger(\vec{p_1},\lambda_1)\hat{a}^\dagger(\vec{p_2},\lambda_2)\left|0\right\rangle = \sqrt{2E_{p_1}}\hat{a}^\dagger(\vec{p_1},\lambda_1)\left|\vec{p_2},\lambda_2\right\rangle =$$
$$= \left|\vec{p_1},\lambda_1;\vec{p_2},\lambda_2\right\rangle = \left|\vec{p_2},\lambda_2;\vec{p_1},\lambda_1\right\rangle = \sqrt{2E_{p_2}}\sqrt{2E_{p_1}}\hat{a}^\dagger(\vec{p_2},\lambda_2)\hat{a}^\dagger(\vec{p_1},\lambda_1)\left|0\right\rangle$$

So:

$$\hat{a}^\dagger(\vec{p_1},\lambda_1)\hat{a}^\dagger(\vec{p_2},\lambda_2)\left|0\right\rangle = \hat{a}^\dagger(\vec{p_2},\lambda_2)\hat{a}^\dagger(\vec{p_1},\lambda_1)\left|0\right\rangle$$

Therefore we have shown

$$\left[\hat{a}^\dagger(\vec{p_1},\lambda_1),\hat{a}^\dagger(\vec{p_2},\lambda_2)\right] = 0$$

For the other we can just take the adjoint of that. ■

Moreover, we have:

$$\left[\hat{a}(\vec{p_1},\lambda_1),\hat{a}^\dagger(\vec{p_2},\lambda_2)\right] = \delta_{\lambda_1\lambda_2}\delta\left(\vec{p_1}-\vec{p_2}\right) \tag{2.76}$$

Proof. On one hand:

$$\sqrt{2E_{p_2}}\hat{a}(\vec{p_1},\lambda_1)\hat{a}^\dagger(\vec{p_2},\lambda_2)\left|\vec{q},\lambda\right\rangle = \hat{a}(\vec{p_1},\lambda_1)\left|\vec{p_2},\lambda_2;\vec{q},\lambda\right\rangle =$$
$$\sqrt{2E_{p_1}}\left(\delta_{\lambda_1\lambda_2}\delta\left(\vec{p_1}-\vec{p_2}\right)\left|\vec{q},\lambda\right\rangle + \delta_{\lambda_1\lambda}\delta\left(\vec{p_1}-\vec{q}\right)\left|\vec{p_2},\lambda_2\right\rangle\right)$$

So we have:

$$\hat{a}(\vec{p_1},\lambda_1)\hat{a}^\dagger(\vec{p_2},\lambda_2)\left|\vec{q},\lambda\right\rangle = \frac{\sqrt{2E_{p_1}}}{\sqrt{2E_{p_2}}}\left(\delta_{\lambda_1\lambda_2}\delta\left(\vec{p_1}-\vec{p_2}\right)\left|\vec{q},\lambda\right\rangle + \delta_{\lambda_1\lambda}\delta\left(\vec{p_1}-\vec{q}\right)\left|\vec{p_2},\lambda_2\right\rangle\right)$$

On the other hand:

$$\hat{a}^\dagger(\vec{p_2},\lambda_2)\hat{a}(\vec{p_1},\lambda_1)\left|\vec{q},\lambda\right\rangle = \hat{a}^\dagger(\vec{p_2},\lambda_2)\sqrt{2E_{p_1}}\delta_{\lambda_1,\lambda}\delta(\vec{p_1}-\vec{q})\left|0\right\rangle$$
$$= \frac{\sqrt{2E_{p_1}}}{\sqrt{2E_{p_2}}}\delta_{\lambda_1\lambda}\delta\left(\vec{p_1}-\vec{q}\right)\left|\vec{p_2},\lambda_2\right\rangle$$

Therefore:

$$\left[\hat{a}(\vec{p_1},\lambda_1),\hat{a}^\dagger(\vec{p_2},\lambda_2)\right]\left|\vec{q},\lambda\right\rangle = \frac{\sqrt{2E_{p_1}}}{\sqrt{2E_{p_2}}}\delta_{\lambda_1\lambda_2}\delta\left(\vec{p_1}-\vec{p_2}\right)\left|\vec{q},\lambda\right\rangle = \delta_{\lambda_1\lambda_2}\delta\left(\vec{p_1}-\vec{p_2}\right)\left|\vec{q},\lambda\right\rangle$$

■

We can introduce the number operator:

$$\hat{N} = \sum_{\lambda=1}^{3}\int d^3p\,\hat{a}^\dagger(\vec{p},\lambda)\hat{a}(\vec{p},\lambda) \tag{2.77}$$

We have (for the following reason we call it "number" operator):

$$\hat{N}\left|\vec{p_1},\lambda_1;\ldots;\vec{p_n},\lambda_n\right\rangle = n\left|\vec{p_1},\lambda_1;\ldots;\vec{p_n},\lambda_n\right\rangle \tag{2.78}$$

Proof. In a similar way as we did in the lectures for hermitian scalar field, it is possible to show that:

$$\left[\hat{N}, \hat{a}(\vec{p}, \lambda)\right] = -\hat{a}(\vec{p}, \lambda) \tag{2.79}$$

$$\left[\hat{N}, \hat{a}^\dagger(\vec{p}, \lambda)\right] = \hat{a}^\dagger(\vec{p}, \lambda) \tag{2.80}$$

Hence, we have:

$$\hat{N} |\vec{p}, \lambda\rangle = \sqrt{2E_p}\hat{N}\hat{a}^\dagger(\vec{p}, \lambda) |0\rangle = \sqrt{2E_p}\left[\hat{N}, \hat{a}^\dagger(\vec{p}, \lambda)\right] |0\rangle = \sqrt{2E_p}\hat{a}^\dagger(\vec{p}, \lambda) |0\rangle = |\vec{p}, \lambda\rangle \tag{2.81}$$

Now, let's start with the induction procedure. We have shown what we want for $n = 1$ and we suppose to be able to do it for $n - 1$:

$$\hat{N} |\vec{p_1}, \lambda_1; \ldots; \vec{p_n}, \lambda_n\rangle = \sqrt{2E_{p_n}}\hat{N}\hat{a}^\dagger(\vec{p}_n, \lambda_n) |\vec{p}_1, \lambda_1; \ldots; \vec{p}_{n-1}, \lambda_{n-1}\rangle = \tag{2.82}$$

$$= \sqrt{2E_{p_n}}\left(\left[\hat{N}, \hat{a}^\dagger(\vec{p}_n, \lambda_n)\right] + \hat{a}^\dagger(\vec{p}_n, \lambda_n)\hat{N}\right) |\vec{p}_1, \lambda_1; \ldots; \vec{p}_{n-1}, \lambda_{n-1}\rangle = \tag{2.83}$$

$$= \sqrt{2E_{p_n}}\left(\hat{a}^\dagger(\vec{p_n}, \lambda_n) + \hat{a}^\dagger(\vec{p}_n, \lambda_n)\,(n-1)\right) |\vec{p}_1, \lambda_1; \ldots; \vec{p}_{n-1}, \lambda_{n-1}\rangle = \tag{2.84}$$

$$= n\,|\vec{p_1}, \lambda_1; \ldots; \vec{p_n}, \lambda_n\rangle \tag{2.85}$$

■

We can introduce the tetramomentum operator:

$$\hat{P}^\mu = \sum_{\lambda=1}^{3} \int d^3q\, q^\mu \hat{a}^\dagger(\vec{q}, \lambda)\hat{a}(\vec{q}, \lambda) \tag{2.86}$$

We have (for the following reason we call it "tetramomentum" operator):

$$\hat{P}^\mu |\vec{p_1}, \lambda_1; \ldots; \vec{p_n}, \lambda_n\rangle = (p_1^\mu + \cdots + p_n^\mu) |\vec{p_1}, \lambda_1; \ldots; \vec{p_n}, \lambda_n\rangle \tag{2.87}$$

Proof. We can show it in analagous way we did in the lectures for hermitian scalar field and in the proof for the equation 2.78 ■

Now, we can define $\hat{A}^\mu(x)$ as a set of operator which act on our Fock space. We can write $\hat{A}^\mu(x)$ as Fourier antitransform:

$$\hat{A}^\mu(\vec{x}) = \int \frac{d^3k}{\sqrt{2E_k}} \hat{q}^\mu(\vec{k}) e^{i\vec{k}\cdot\vec{x}} \tag{2.88}$$

Considering that we deal with real vector field and the fact that we have 3 physical degrees of freedom, we can write $\hat{q}^\mu$ such that (as done for hermitial scalar field):

$$\hat{A}^\nu(x) = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2E_\mathbf{k}}} \sum_{\lambda=1}^{3} \left(\hat{a}(\vec{k}, \lambda)\epsilon^\nu(\vec{k}, \lambda)e^{-ikx} + \hat{a}^\dagger(\vec{k}, \lambda)\epsilon^{\nu*}(\vec{k}, \lambda)e^{ikx}\right) \tag{2.89}$$

In the last equation we have used the tetravectors $\epsilon^\nu(\vec{k}, \lambda)$ defined before and moreover we moved to Heisenberg picture, hence we have a field operator depending by time (the transformation from Schrödinger to Heisenberg picture of the fields could be shown in the same way we did in the lectures for hermitian scalar field). It must be pointed out that the expression 2.89 is the quantized field of the classical one written in eq(2.56) where $a(\vec{k},\lambda)$ and $a^*(\vec{k},\lambda)$ are promoted to be operators.

We can state that A^ν satisfies:

$$\left(\Box + m^2\right)\hat{A}^\nu = 0 \tag{2.90}$$

since A^ν in eq 2.89 is of the same form of the solutions of the previous equation .

These equations of motion could come from the quantized density of our Lagrangian:

$$\hat{\mathcal{L}} = N\left(-\frac{1}{2}\partial^\mu\hat{A}^\nu\partial_\mu\hat{A}_\nu + \frac{1}{2}\partial^\mu\hat{A}^\nu\partial_\nu\hat{A}_\mu + \frac{1}{2}m^2\hat{A}_\mu\hat{A}^\mu\right) \tag{2.91}$$

where $N(\dots)$ indicates the normal ordering product used to avoid divergent contributions that could appear for istance when we compute the hamiltonian.

The physical interpretation to apply $|0\rangle$ to $\hat{A}^\nu(x)$ (see eq 2.89) is to produce a superposition of single particle states which propagates in space time:

$$\hat{A}^\nu(x)\,|0\rangle = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2E_{\mathbf{k}}}}\sum_{\lambda=1}^{3}\left(\hat{a}^\dagger(\vec{k},\lambda)\epsilon^{\nu*}(\vec{k},\lambda)\right)e^{ikx}\,|0\rangle = \tag{2.92}$$

$$= \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}2E_{\mathbf{k}}}\sum_{\lambda=1}^{3}\epsilon^{\nu*}(\vec{k},\lambda)e^{ikx}\left|\vec{k},\lambda\right\rangle \tag{2.93}$$

To make more evident what we stated before we can apply a ket of single particle state $\langle\vec{p},r|$ to the left of previous equation:

$$\langle\vec{p},r|\,\hat{A}^\nu(x)\,|0\rangle = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}2E_{\mathbf{k}}}\sum_{\lambda=1}^{3}\epsilon^{\nu*}(\vec{k},\lambda)e^{ikx}\left\langle\vec{p},r\middle|\vec{k},\lambda\right\rangle = \tag{2.94}$$

$$= \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}2E_{\mathbf{k}}}\sum_{\lambda=1}^{3}\epsilon^{\nu*}(\vec{k},\lambda)e^{ikx}2E_p\delta_{r\lambda}\delta(\vec{p}-\vec{k}) \tag{2.95}$$

$$= \epsilon^{\nu*}(\vec{p},r)\frac{e^{ipx}}{(2\pi)^{\frac{3}{2}}} \tag{2.96}$$

So

$$\langle\vec{p},r|\,\hat{A}^\nu(x)\,|0\rangle = \epsilon^{\nu*}(\vec{p},r)\frac{e^{ipx}}{(2\pi)^{\frac{3}{2}}} \tag{2.97}$$

We can easily obtain taking the complex conjugate also :

$$\langle 0|\,\hat{A}^\nu(x)\,|\vec{p},r\rangle = \epsilon^{\nu}(\vec{p},r)\frac{e^{-ipx}}{(2\pi)^{\frac{3}{2}}} \tag{2.98}$$

Therefore, each incoming vector particle with spin 1 contributes a factor of $\epsilon^\nu(\vec{p},r)$ to the amplitude in addition to the usual exponential factor.

Equations $2.97, 2.98$ lead to the Feynman rules in momentum space:

For every incoming (outgoing) particle of this kind with momentum p and polarization r, we include a factor $\frac{\epsilon^\nu(\vec{p},r)}{(2\pi)^{\frac{3}{2}}}\left(\frac{\epsilon^{\nu*}(\vec{p},r)}{(2\pi)^{\frac{3}{2}}}\right)$.

We define the "Positive frequency evolution kernel" and "Negative frequency evolution kernel" (as we did for hermitian scalar field):

$$\Delta^+(x) = \frac{1}{(2\pi)^3}\int \frac{d^3p}{2E_p}e^{-ipx}\bigg|_{E_p=p^0} \tag{2.99}$$

$$\Delta^-(x) = \frac{1}{(2\pi)^3}\int \frac{d^3p}{2E_p}e^{ipx}\bigg|_{E_p=p^0} \tag{2.100}$$

We are going to show that:

$$\langle 0|\,\hat{A}^\nu(x)\hat{A}^\mu(y)\,|0\rangle = \left(-\eta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{m^2}\right)\Delta^+(x-y) \tag{2.101}$$

where derivatives act on $(x-y)$.

Proof. We have to compute the product $\langle 0|\,\hat{A}^\nu(x)\hat{A}^\mu(y)\,|0\rangle$ with:

$$\hat{A}^\nu(x) = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2E_{\mathbf{k}}}}\sum_{\lambda=1}^{3}\left(\hat{a}(\vec{k},\lambda)\epsilon^\nu(\vec{k},\lambda)e^{-ikx} + \hat{a}^\dagger(\vec{k},\lambda)\epsilon^{\nu*}(\vec{k},\lambda)e^{ikx}\right) \tag{2.102}$$

and we observe that the only term that survives is the one which include $\hat{a}(\vec{k},\lambda)\hat{a}^\dagger(\vec{k},\lambda)$:

$$\langle 0|\,\hat{A}^\nu(x)\hat{A}^\mu(y)\,|0\rangle = \tag{2.103}$$

$$= \langle 0|\int \frac{d^3k\,d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}\sqrt{2E_{\mathbf{k}}}}\sum_{\lambda,r=1}^{3}\left(\hat{a}(\vec{k},\lambda)\hat{a}^\dagger(\vec{p},r)\epsilon^\nu(\vec{k},\lambda)\epsilon^{\nu*}(\vec{p},r)e^{-ikx}e^{ipy}\right)|0\rangle = \tag{2.104}$$

$$= \langle 0|\int \frac{d^3k\,d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}\sqrt{2E_{\mathbf{k}}}}\sum_{\lambda,r=1}^{3}\left(\left[\hat{a}(\vec{k},\lambda),\hat{a}^\dagger(\vec{p},r)\right]\epsilon^\nu(\vec{k},\lambda)\epsilon^{\nu*}(\vec{p},r)e^{-ikx}e^{ipy}\right)|0\rangle = \tag{2.105}$$

$$= \langle 0|\int \frac{d^3k\,d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}\sqrt{2E_{\mathbf{k}}}}\sum_{\lambda,r=1}^{3}\left(\delta_{\lambda r}\delta\left(\vec{k}-\vec{p}\right)\epsilon^\nu(\vec{k},\lambda)\epsilon^{\nu*}(\vec{p},r)e^{-ikx}e^{ipy}\right)|0\rangle = \tag{2.106}$$

$$= \langle 0|\int \frac{d^3k}{(2\pi)^3 2E_{\mathbf{k}}}\sum_{r=1}^{3}\epsilon^\nu(\vec{k},r)\epsilon^{\nu*}(\vec{k},r)e^{-ikx}e^{iky}\,|0\rangle \tag{2.107}$$

$$\tag{2.108}$$

where we used the fact that $p^0 = \sqrt{E_p + m^2}$ and $k^0 = \sqrt{E_k + m^2}$. So, using our completeness relation:

$$\langle 0|\,\hat{A}^\nu(x)\hat{A}^\mu(y)\,|0\rangle = \tag{2.109}$$

$$= \langle 0|\int \frac{d^3k}{(2\pi)^3 2E_{\mathbf{k}}}\left(-\eta^{\mu\nu} + \frac{k^\mu k^\nu}{m^2}\right)e^{ik(y-x)}\,|0\rangle = \tag{2.110}$$

$$= \langle 0|\left(-\eta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{m^2}\right)\int \frac{d^3k}{(2\pi)^3 2E_{\mathbf{k}}}e^{-ik(x-y)}\,|0\rangle = \tag{2.111}$$

$$= \left(-\eta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{m^2}\right)\Delta^+(x-y) \tag{2.112}$$

■

$$\langle 0|\,\hat{A}^\nu(y)\hat{A}^\mu(x)\,|0\rangle = \left(-\eta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{m^2}\right)\Delta^-(x-y) \tag{2.113}$$

Proof.

$$\langle 0| \hat{A}^\nu(y)\hat{A}^\mu(x) |0\rangle = \left(-\eta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{m^2}\right)\Delta^+(y-x) = \left(-\eta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{m^2}\right)\Delta^-(x-y) \quad (2.114)$$

where we have used that:

$$\Delta^+(-x) = \Delta^-(x) \quad (2.115)$$

■

From the previous relation we can get:

$$\langle 0| \left[\hat{A}^\nu(x), \hat{A}^\mu(y)\right] |0\rangle = \left(-\eta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{m^2}\right)\Delta(x-y) \quad (2.116)$$

where

$$\Delta(x) = \Delta^+(x) - \Delta^-(x) \quad (2.117)$$

The physical meaning of $\langle 0| \hat{A}^\nu(x)\hat{A}^\mu(y) |0\rangle$ is the probability that a particle produced at y could be detected at x. We observe that if x and y are such that $(x-y)^2 < 0$, $\langle 0| \hat{A}^\nu(x)\hat{A}^\mu(y) |0\rangle = \left(-\eta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{m^2}\right)\Delta^+(x-y) \neq 0$.

At first glance, it could seem a possible violation of causality because if $(x-y)^2 < 0$ the ordering time is not Lorentz invariant. But this is not the case because field operators are not observables which are at least bilinear form of the fields. Hence, if $\hat{O}_1(x)$ and $\hat{O}_2(y)$ are observables, the commutator $\left[\hat{O}_1(x), \hat{O}_2(y)\right]$ could be expressed as a sum of terms all containing the commutator between two fields.

So we are not interested in $\langle 0| \hat{A}^\nu(x)\hat{A}^\mu(y) |0\rangle$ but in $\langle 0| \left[\hat{A}^\nu(x), \hat{A}^\mu(y)\right] |0\rangle$ which, if $(x-y)$ is spacelike ($(x-y)^2 < 0$), is 0 due to eq. (2.116) and to the property of $\Delta(x-y)$ we mentioned in the lecture for the hermitian scalar field.

Now we define the Feynman propagator:

$$D_F^{\mu\nu}(x-y) \equiv \langle 0| T\left(\hat{A}^\nu(x)\hat{A}^\mu(y)\right) |0\rangle = \quad (2.118)$$

$$= \theta(x^0-y^0)\langle 0| \hat{A}^\nu(x)\hat{A}^\mu(y) |0\rangle + \theta(y^0-x^0)\langle 0| \hat{A}^\nu(y)\hat{A}^\mu(x) |0\rangle = \quad (2.119)$$

$$= \theta(x^0-y^0)\left(-\eta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{m^2}\right)\Delta^+(x-y) + \theta(y^0-x^0)\left(-\eta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{m^2}\right)\Delta^-(x-y) = \quad (2.120)$$

$$= \left(-\eta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{m^2}\right)\left(\theta(x^0-y^0)\Delta^+(x-y) + \theta(y^0-x^0)\Delta^-(x-y)\right) \quad (2.121)$$

For the hermitian scalar field we saw that:

$$\theta(x^0-y^0)\Delta^+(x-y) + \theta(y^0-x^0)\Delta^-(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{i}{k^2-m^2+i\epsilon} \quad (2.122)$$

Hence we have:

$$D_F^{\mu\nu}(x-y) = \left(-\eta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{m^2}\right)\int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{i}{k^2-m^2+i\epsilon} = \quad (2.123)$$

$$= \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{i\left(-\eta^{\mu\nu} + \frac{k^\mu k^\nu}{m^2}\right)}{k^2-m^2+i\epsilon} \quad (2.124)$$

$$(2.125)$$

This leads to the propagator for a massive vector field:

$$\tilde{D}_F^{\mu\nu}(k) = \frac{i\left(-\eta^{\mu\nu} + \frac{k^\mu k^\nu}{m^2}\right)}{k^2 - m^2 + i\epsilon} \tag{2.126}$$

The derivation was not quite right when $\mu = \nu = 0$ in the equality 2.121 . In this case, the time derivatives don't commute with the θ functions and there is the additional term when one of the derivatives acts on the θ function, giving a factor of $\delta(x^0 - y^0)$, and the other acts on the Δ^+ function. This wasn't a problem in the Dirac spinor case because there was only a single time derivative, and the term vanished because $\Delta(x-y) = 0$. In this case, it doesn't happen, hence there is an additional term. In the path integral formulation of quantum field theory this problem doesn't arise. The previous proof could be taken as valid for $(\mu, \nu) \neq (0,0)$, and we can argue that by Lorentz invariance the result must have this form for $(\mu, \nu) = (0,0)$ as well.

We could have followed a canonical quantization approach to quantize the theory: we could have promoted the field A^i and its conjugate momentum $\Pi^i_A = E^i$ (as stated in 2.57) to operators, satisfying the canonical commutation relations:

$$\left[\hat{A}^i(x), \hat{A}^j(y)\right]_{x^0=y^0} = \left[\hat{E}^i(x), \hat{E}^j(y)\right]_{x^0=y^0} = 0 \tag{2.127}$$

$$\left[\hat{A}_i(x), \hat{E}^j(y)\right]_{x^0=y^0} = i\delta_i^j \delta\left(\vec{x} - \vec{y}\right) \tag{2.128}$$

and from that we could get the commutation relation of creation and annihilation operator from the expression of the field and its conjugate momentum .

We can quantize the Lagrangian (as in expression 2.91), from which we can obtain the Energy-Momentum Tensor and therefore the total momentum operator P and the total energy E (performing a space integration).

Putting into these expressions the one for $\hat{A}^\nu(x)$, we can recover the result we had in eq2.86.

3 Exercise 3: decay of a massive scalar particle

Consider two types of neutral, scalar particles ξ and η of masses $0 < m_\xi < m_\eta$. The two particles are described by two scalar, Hermitian fields $\hat{\phi}$ and $\hat{\Phi}$ with a density of Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\hat{\phi}\partial^\mu\hat{\phi} - \frac{1}{2}m_\xi^2\hat{\phi}^2 + \frac{1}{2}\partial_\mu\hat{\Phi}\partial^\mu\hat{\Phi} - \frac{1}{2}m_\eta^2\hat{\Phi}^2 - \mu\hat{\Phi}\hat{\phi}^2$$

This lagrangian allows the decay of a particle η into ξ particles.

- Find the dimensions (in natural units) of the coupling constant μ and n_{min} , the minimum amount of ξ particles in the final state [2*pt*]
- Find the lowest order contribution in μ to the matrix element of the transition matrix $i\hat{T}$ for the decay of a η particle into n_{min} ξ particles [4pt]; suggestion: remembering the LSZ formula, if you want, you can drop the $(2\pi)^{9/2}$ factor that comes out from the direct calculations.
- find the conditions on the particle masses such that the decay in n_{min} particles is kinematically allowed [1pt];
- under these conditions, compute the lifetime of a η particle at lowest order in μ[4pt].

The lifetime is defined as the inverse of the decay rate Γ. The generic formula for the differential decay rate for one particle η decaying into $n\,\xi$ particles is very similar to the one of the differential cross section:

$$\mathrm{d}\Gamma = \frac{S}{2m_\eta}\left(\mathrm{d}\Pi_n\right)\left|\mathcal{M}\left(\eta \to \xi_1, \ldots, \xi_n\right)\right|^2 \tag{3.1}$$

where $(\mathrm{d}\Pi_n)$ is the invariant n -particle final state:

$$(\mathrm{d}\Pi_n) = \prod_{i=1,n}\frac{d^3p_i}{(2\pi)^3}\frac{1}{2E_i}(2\pi)^4\delta_{(4)}\left(p_\eta - \sum_{i=1,n}p_i\right)$$

while S is the symmetry factor for identical particles in the final state:

$$S = \prod_{j=1,m}\frac{1}{n_j!}$$

where the index j runs over all m kinds of different particles in the final state and n_j is the number of identical particles j in the final state, such that: $\sum_{j=1,m} n_j = n$

Solution

The two particles are described by two scalar, Hermitian fields $\hat{\phi}$ and $\hat{\Phi}$ with a density of Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\hat{\phi}\partial^\mu\hat{\phi} - \frac{1}{2}m_\xi^2\hat{\phi}^2 + \frac{1}{2}\partial_\mu\hat{\Phi}\partial^\mu\hat{\Phi} - \frac{1}{2}m_\eta^2\hat{\Phi}^2 - \mu\hat{\Phi}\hat{\phi}^2 \tag{3.2}$$

Throughout this exercise we will work with "natural" units, defined by

$$\hbar = c = 1 \tag{3.3}$$

which allows us to express all dimensionful quantities in terms of a single scale which we choose to be mass or, equivalently, energy (since $E = mc^2$ has become $E = m$). To convert the unit of energy back to a unit of length or time, we need to insert the relevant powers of c and $\hbar$. Since the length scale λ associated to a mass m is the Compton wavelength:

$$\lambda = \frac{\hbar}{mc} \tag{3.4}$$

we have:

$$[\text{Lenght}] = M^{-1} \tag{3.5}$$

and from $x = ct$, we get:

$$[\text{Time}] = M^{-1} \tag{3.6}$$

We are going to show that coupling constant μ which appears in the lagrangian density $\mathcal{L}$ of eq(3.2) has dimension of a mass:

$$[\mu] = M \tag{3.7}$$

Proof.

1. $[\mathcal{L}] = M^4$.

   Subproof.

   Firstly, note that the action S ($[S] = [\int L\, dt] = [E\, dt] = [\hbar]$ for uncertainty relation) has dimensions of angular momentum or, equivalently, the same dimensions as $\hbar$. Since we've set $\hbar = 1$, we have $[S] = 1$. With $S = \int d^4x\, \mathcal{L}$, and $[d^4x] = [d^3x dt] = M^{-4}$, the Lagrangian density must therefore have dimension M^4. ■

2. $[\Phi] = [\phi] = M$.

   Subproof. Since lagrangian density has dimension of M^4, also each term in eq3.2 has dimension M^4. In particular also the term $[\frac{1}{2} m_\eta^2 \hat{\Phi}^2] = M^4$. Hence

   $$M^4 = [m_\eta^2][\hat{\Phi}^2] = M^2[\hat{\Phi}^2] \tag{3.8}$$

   Hence $[\hat{\Phi}^2] = M^2$. The same argument could be used for ϕ ■

Since lagrangian density has dimension of M^4, also the term $[\mu\hat{\Phi}\hat{\phi}^2] = M^4$:

$$M^4 = [\mu][\hat{\Phi}][\hat{\phi}^2] = [\mu]M^3 \tag{3.9}$$

Hence: $[\mu] = M$ ■

We can decompose our lagrangian into :

$$\hat{\mathcal{L}} = \hat{\mathcal{L}}_{0_\Phi} + \hat{\mathcal{L}}_{0_\phi} + \hat{\mathcal{L}}_{\text{int}} \tag{3.10}$$

where:

$$\hat{\mathcal{L}}_{0_\Phi} = \frac{1}{2}\partial_\mu\hat{\Phi}\partial^\mu\hat{\Phi} - \frac{1}{2}m_\eta^2\hat{\Phi}^2 \tag{3.11}$$

$$\hat{\mathcal{L}}_{0_\phi} = \frac{1}{2}\partial_\mu\hat{\phi}\partial^\mu\hat{\phi} - \frac{1}{2}m_\xi^2\hat{\phi}^2 \tag{3.12}$$

$$\hat{\mathcal{L}}_{\text{int}} = -\mu\hat{\Phi}\hat{\phi}^2 \tag{3.13}$$

We have:

$$\hat{H} = \hat{H}_0 + \hat{V} \tag{3.14}$$

where $\hat{H}_0$ is the free hamiltonian while $\hat{V}(t) = \int d^3x \hat{\mathcal{U}}(x)$ is the interaction hamiltonian.

Since $\mathcal{L}_{\text{int}}$ does depend only on the fields and not on the derivatives of the fields, then :

$$\hat{\mathcal{L}}_{\text{int}} = -\hat{\mathcal{U}} = \mu \hat{\Phi} \hat{\phi}^2 \tag{3.15}$$

As seen in the lectures, we have the $\hat{S}$ matrix:

$$\hat{S} = \hat{U}_I(+\infty, -\infty) \tag{3.16}$$

where:

$$\hat{U}_I(+\infty, -\infty) = \lim_{\substack{\tau \to +\infty \\ \tau_0 \to -\infty}} \exp\left(iH_0\tau\right) \exp\left(-iH\left(\tau - \tau_0\right)\right) \exp\left(-iH_0\tau_0\right) \tag{3.17}$$

We can show that:

$$\hat{S} = \hat{U}_I(+\infty, -\infty) = \mathbb{1} + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} d^4x_1 \cdots d^4x_n T\left\{\hat{\mathcal{U}}_I(x_1) \cdots \hat{\mathcal{U}}_I(x_n)\right\} = \tag{3.18}$$

$$= T\left(\exp\left(-i \int_{-\infty}^{+\infty} d^4x \hat{\mathcal{U}}_I(x)\right)\right) = \sum_{n=0}^{\infty} \hat{S}^{(n)} \tag{3.19}$$

where:

$$\hat{\mathcal{U}}_I(t, \vec{x}) = \hat{U}_0^\dagger(t) \mathcal{U}_s(\vec{x}) \hat{U}_0(t) \tag{3.20}$$

where $\mathcal{U}_s(\vec{x})$ is the interaction potential operator in the Schrodinger picture and:

$$\hat{U}_0(t) = exp(-i\hat{H}_0 t) \tag{3.21}$$

In our case:

$$\hat{\mathcal{U}}_I(x) = \mu \hat{\Phi}_I(x) \hat{\phi}_I^2(x) \tag{3.22}$$

Proof.

$$\hat{\mathcal{U}}_I(x) = \hat{U}_0^\dagger(t) \mathcal{U}_s(\vec{x}) \hat{U}_0(t) = \mu \hat{U}_0^\dagger(t) \hat{\Phi}_s(x) \hat{\phi}_s(x) \hat{\phi}_s(x) \hat{U}_0(t) = \tag{3.23}$$

$$= \mu \hat{U}_0^\dagger(t) \hat{\Phi}_s(x) \hat{U}_0(t) \hat{U}_0^\dagger(t) \hat{\phi}_s(x) \hat{U}_0(t) \hat{U}_0^\dagger(t) \hat{\phi}_s(x) \hat{U}_0(t) = \tag{3.24}$$

$$= \mu \hat{\Phi}_I(x) \hat{\phi}_I(x) \hat{\phi}_I(x) \tag{3.25}$$

■

Since the fields in the interaction picture of an interacting theory are identical to the fields in the Heisenberg picture of the corresponding free theory, we have:

$$\hat{\Phi}_I(x) = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}} \sqrt{2E_{\mathbf{k}}}} \left(\hat{a}(\vec{k}) e^{-ikx} + \hat{a}^\dagger(\vec{k}) e^{ikx}\right) \tag{3.26}$$

$$\hat{\phi}_I(x) = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}} \sqrt{2E_{\mathbf{k}}}} \left(\hat{b}(\vec{k}) e^{-ikx} + \hat{b}^\dagger(\vec{k}) e^{ikx}\right) \tag{3.27}$$

Moreover:

$$S_{\beta\alpha} \equiv \langle \phi_\beta | \hat{S} | \phi_\alpha \rangle = \delta(\alpha - \beta) + i \langle \phi_\beta | \hat{T} | \phi_\alpha \rangle = \delta(\alpha - \beta) + (2\pi)^4 \delta_4 \left(p_\alpha - p_\beta\right) i \mathcal{M}_{\alpha\beta} \tag{3.28}$$

In the decay of a particle η into ξ particles we want to find $n_{\min}$, the minimum amount of ξ particles in the final state .

Since the physical interpretation of $S_{\beta\alpha}$ is that its square modulus is proportional to the transition probabilty between two different states, we can find the n_{min} value for which we get a contribution of $S_{\beta\alpha}$ different from 0. .

Our initial state is:

$$|\phi_\alpha\rangle = \left|\eta : \vec{k}\right\rangle = \sqrt{2E_k}\hat{a}^\dagger(\vec{k})\,|0\rangle \tag{3.29}$$

while our final state :

$$|\phi_\beta\rangle = |\xi_1 : \vec{p_1}; \ldots ; \xi_{n_{\min}} : \vec{p}_{n_{\min}}\rangle = \sqrt{2E_{p_1}\cdots 2E_{p_n}}\hat{b}^\dagger(\vec{p}_1)\cdots\hat{b}^\dagger(\vec{p}_{n_{\min}})\,|0\rangle \tag{3.30}$$

The delta contribution of $S_{\beta\alpha}$, which appears in eq. 3.28, vanishes because we have two different particles in the initial and final states. Starting from the first order in μ in the expansion of eq 3.19:

$$-\,i\,\langle\phi_\beta|\int_{-\infty}^{+\infty} d^4x\,T\left(\hat{\mathcal{U}}_I(x)\right)|\phi_\alpha\rangle = -i\mu\,\langle\phi_\beta|\int_{-\infty}^{+\infty} d^4x\,T\left(\hat{\Phi}_I(x)\hat{\phi}_I^2(x)\right)|\phi_\alpha\rangle = \tag{3.31}$$

$$= -i\mu\sqrt{2E_{p_1}\cdots 2E_{p_n}}\sqrt{2E_k}\int_{-\infty}^{+\infty} d^4x\,\langle 0|\,\hat{b}(\vec{p}_1)\cdots\hat{b}(\vec{p}_{n_{\min}})T\left(\hat{\Phi}_I(x)\hat{\phi}_I^2(x)\right)\hat{a}^\dagger(\vec{k})\,|0\rangle \tag{3.32}$$

Applying Wick theorem we have:

$$T\left(\hat{\Phi}_I(x)\hat{\phi}_I^2(x)\right) =: \hat{\Phi}_I(x)\hat{\phi}_I(x)\hat{\phi}_I(x) : + : \hat{\Phi}_I(x) : \overbrace{\phi_x\phi_x} \tag{3.33}$$

Let's consider only the operator part in the previous equation to make calculation 3.32 easier, which means $\Phi_I \propto \hat{a} + \hat{a}^\dagger$ and $\phi_I \propto \hat{b} + \hat{b}^\dagger$ (it must be pointed out that the used "proportional symbol" $\propto$ means that the LHS is proportional to RHS if you get rid of all pieces that are not operators). Starting from the first term in eq(3.33):

$$\langle 0|\,\hat{b}(\vec{p}_1)\cdots\hat{b}(\vec{p}_{n_{\min}}) : (\hat{a}+\hat{a}^\dagger)(\hat{b}+\hat{b}^\dagger)(\hat{b}+\hat{b}^\dagger) : \hat{a}^\dagger(\vec{k})\,|0\rangle \tag{3.34}$$

In order to not get 0, we need $n_{min} = 2$. Considering the second term in eq(3.33) in the calculation 3.32, it vanishes :

$$\langle 0|\,\hat{b}(\vec{p}_1)\cdots\hat{b}(\vec{p}_{n_{\min}}) : (\hat{a}+\hat{a}^\dagger) : \hat{a}^\dagger(\vec{k})\,|0\rangle \tag{3.35}$$

in fact for each possible n_{min} the operators $\hat{b}$ give 0 acting on $|0\rangle$. Hence, we observe that we get a contribution different from 0 if $n_{min} = 2$.

Moreover, there is no term in the expansion of S matrix that give a contribution different from 0 in the case $n_{min} = 1$ in fact from previous calculation we see that the first order of S matrix element vanishes and if we repeat the argument for the next orders in μ we can show easily that they also vanish.

Hence $n_{min} = 2$.

Let's compute explicitly the lowest order contribution in μ to the matrix element of the transition matrix. We go on the calculation 3.32 with $n_{min} = 2$. Before, we showed that the only term that doesn't vanish in eq 3.33 is the first. So putting that into 3.32, we get:

$$\langle\phi_\beta|\,\hat{S}^{(1)}\,|\phi_\alpha\rangle = -i\mu\sqrt{2E_{p_1}2E_{p_2}}\sqrt{2E_k}\int_{-\infty}^{+\infty} d^4x\,\langle 0|\,\hat{b}(\vec{p}_1)\hat{b}(\vec{p}_2) : \hat{\Phi}_I(x)\hat{\phi}_I(x)\hat{\phi}_I(x) : \hat{a}^\dagger(\vec{k})\,|0\rangle = \tag{3.36}$$

$$= -i\mu\sqrt{2E_{p_1}2E_{p_2}}\sqrt{2E_k}\int_{-\infty}^{+\infty} d^4x\,\langle 0|\,\hat{b}(\vec{p}_1)\hat{b}(\vec{p}_2) : \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}\sqrt{2E_{\mathbf{p}}}}\left(\hat{a}(\vec{p})e^{-ipx} + \hat{a}^\dagger(\vec{p})e^{ipx}\right) \tag{3.37}$$

$$\int \frac{d^3q}{(2\pi)^{\frac{3}{2}}\sqrt{2E_{\mathbf{q}}}}\left(\hat{b}(\vec{q})e^{-iqx} + \hat{b}^\dagger(\vec{q})e^{iqx}\right)\int \frac{d^3l}{(2\pi)^{\frac{3}{2}}\sqrt{2E_{\mathbf{l}}}}\left(\hat{b}(\vec{l})e^{-ilx} + \hat{b}^\dagger(\vec{l})e^{ilx}\right) : \hat{a}^\dagger(\vec{k})\,|0\rangle \tag{3.38}$$

(3.39)

Let's considering only the operator part.

Performing the product of the operators, we have 8 different terms (considering the normal ordering product):

$$\hat{a}(\vec{p})\hat{b}(\vec{q})\hat{b}(\vec{l}),\ \hat{b}^\dagger(\vec{l})\hat{a}(\vec{p})\hat{b}(\vec{q}),\ \hat{b}^\dagger(\vec{q})\hat{a}(\vec{p})\hat{b}(\vec{l}),\ \hat{b}^\dagger(\vec{q})\hat{b}^\dagger(\vec{l})\hat{a}(\vec{p}), \tag{3.40}$$

$$\hat{a}^\dagger(\vec{p})\hat{b}(\vec{q})\hat{b}(\vec{l}),\ \hat{b}^\dagger(\vec{l})\hat{a}^\dagger(\vec{p})\hat{b}(\vec{q}),\ \hat{a}^\dagger(\vec{p})\hat{b}^\dagger(\vec{q})\hat{b}(\vec{l}),\ \hat{a}^\dagger(\vec{p})\hat{b}^\dagger(\vec{q})\hat{b}^\dagger(\vec{l}), \tag{3.41}$$

(3.42)

and the only one that survives is the one that could "annihilate" a η particle in the initial state and two particles ξ in the final states: $\hat{b}^\dagger(\vec{q})\hat{b}^\dagger(\vec{l})\hat{a}(\vec{p})$.

In a more explicit way, we could do the brute force computation and show through the commutation relations that all the terms vanish except this one.

$$\langle\phi_\beta|\,\hat{S}^{(1)}\,|\phi_\alpha\rangle =$$

$$= -i\mu\sqrt{2E_{p_1}2E_{p_2}}\sqrt{2E_k}\int_{-\infty}^{+\infty} d^4x\,\langle 0|\,\hat{b}(\vec{p}_1)\hat{b}(\vec{p}_2) : \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}\sqrt{2E_{\mathbf{p}}}}\left(\hat{a}(\vec{p})e^{-ipx}\right)$$

$$\int \frac{d^3q}{(2\pi)^{\frac{3}{2}}\sqrt{2E_{\mathbf{q}}}}\left(\hat{b}^\dagger(\vec{q})e^{iqx}\right)\int \frac{d^3l}{(2\pi)^{\frac{3}{2}}\sqrt{2E_{\mathbf{l}}}}\left(\hat{b}^\dagger(\vec{l})e^{ilx}\right) : \hat{a}^\dagger(\vec{k})\,|0\rangle =$$

$$= -\frac{i\mu\sqrt{2E_{p_1}2E_{p_2}2E_k}}{(2\pi)^{\frac{9}{2}}}\int_{-\infty}^{+\infty} d^4x\,\langle 0|\,\hat{b}(\vec{p}_1)\hat{b}(\vec{p}_2)\int \frac{d^3p\,d^3q\,d^3l}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{l}}2E_{\mathbf{q}}}}\hat{b}^\dagger(\vec{q})\hat{b}^\dagger(\vec{l})\hat{a}(\vec{p})e^{-i(p-q-l)x}\hat{a}^\dagger(\vec{k})\,|0\rangle =$$

$$= -\frac{i\mu\sqrt{2E_{p_1}2E_{p_2}2E_k}}{(2\pi)^{\frac{9}{2}}}\int_{-\infty}^{+\infty} d^4x\int \frac{d^3p\,d^3q\,d^3l}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{l}}2E_{\mathbf{q}}}}e^{-i(p-q-l)x}\,\langle 0|\,\hat{b}(\vec{p}_1)\hat{b}(\vec{p}_2)\hat{b}^\dagger(\vec{q})\hat{b}^\dagger(\vec{l})\hat{a}(\vec{p})\hat{a}^\dagger(\vec{k})\,|0\rangle$$

Let's evaluate the operator part (using the commutation relations):

$$\langle 0|\,\hat{b}(\vec{p}_1)\hat{b}(\vec{p}_2)\hat{b}^\dagger(\vec{q})\hat{b}^\dagger(\vec{l})\hat{a}(\vec{p})\hat{a}^\dagger(\vec{k})\,|0\rangle = \tag{3.43}$$

$$= \langle 0|\,\hat{b}(\vec{p}_1)\left(\left[\hat{b}(\vec{p}_2),\hat{b}^\dagger(\vec{q})\right] + \hat{b}^\dagger(\vec{q})\hat{b}(\vec{p}_2)\right)\hat{b}^\dagger(\vec{l})\left[\hat{a}(\vec{p}),\hat{a}^\dagger(\vec{k})\right]|0\rangle = \tag{3.44}$$

$$= \langle 0|\,\hat{b}(\vec{p}_1)\left(\delta(\vec{p}_2-\vec{q}) + \hat{b}^\dagger(\vec{q})\hat{b}(\vec{p}_2)\right)\hat{b}^\dagger(\vec{l})\delta(\vec{p}-\vec{k})\,|0\rangle = \tag{3.45}$$

$$= \delta(\vec{p}_2-\vec{q})\delta(\vec{p}-\vec{k})\,\langle 0|\,\hat{b}(\vec{p}_1)\hat{b}^\dagger(\vec{l})\,|0\rangle + \delta(\vec{p}-\vec{k})\,\langle 0|\,\hat{b}(\vec{p}_1)\hat{b}^\dagger(\vec{q})\hat{b}(\vec{p}_2)\hat{b}^\dagger(\vec{l})\,|0\rangle = \tag{3.46}$$

$$= \delta(\vec{p}_2-\vec{q})\delta(\vec{p}-\vec{k})\delta(\vec{p}_1-\vec{l}) + \delta(\vec{p}-\vec{k})\delta(\vec{p}_1-\vec{q})\delta(\vec{p}_2-\vec{l}) \tag{3.47}$$

(3.48)

The fact that we have two terms was predictable because we had only a way to annihilate the η particle in the initial state, while we had 2 ways to annihilate the two ξ particles in the final state.

Let's substitute what we have just computed:

$$\begin{aligned}
&\langle\phi_\beta|\,\hat{S}^{(1)}\,|\phi_\alpha\rangle = \\
&= -\frac{i\mu\sqrt{2E_{p_1}2E_{p_2}2E_k}}{(2\pi)^{\frac{9}{2}}}\int_{-\infty}^{+\infty} d^4x \int \frac{d^3p\,d^3q\,d^3l}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{l}}2E_{\mathbf{q}}}} e^{-i(p-q-l)x}\delta(\vec{p}_2-\vec{q})\delta(\vec{p}-\vec{k})\delta(\vec{p}_1-\vec{l}) \\
&- \frac{i\mu\sqrt{2E_{p_1}2E_{p_2}2E_k}}{(2\pi)^{\frac{9}{2}}}\int_{-\infty}^{+\infty} d^4x \int \frac{d^3p\,d^3q\,d^3l}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{l}}2E_{\mathbf{q}}}} e^{-i(p-q-l)x}\delta(\vec{p}-\vec{k})\delta(\vec{p}_1-\vec{q})\delta(\vec{p}_2-\vec{l}) = \\
&= -\frac{i\mu}{(2\pi)^{\frac{9}{2}}}\int d^3p\,d^3q\,d^3l(2\pi)^4\delta(p-q-l)\delta(\vec{p}_2-\vec{q})\delta(\vec{p}-\vec{k})\delta(\vec{p}_1-\vec{l}) \\
&- \frac{i\mu}{(2\pi)^{\frac{9}{2}}}\int d^3p\,d^3q\,d^3l(2\pi)^4\delta(p-q-l)\delta(\vec{p}-\vec{k})\delta(\vec{p}_1-\vec{q})\delta(\vec{p}_2-\vec{l}) = \\
&= -\frac{i\mu}{(2\pi)^{\frac{9}{2}}}(2\pi)^4\delta(k-p_2-p_1) - \frac{i\mu}{(2\pi)^{\frac{9}{2}}}(2\pi)^4\delta(k-p_2-p_1) \\
&= -\frac{2i\mu}{(2\pi)^{\frac{9}{2}}}(2\pi)^4\delta(k-p_2-p_1)
\end{aligned}$$

Therefore:

$$\langle\phi_\beta|\,\hat{S}^{(1)}\,|\phi_\alpha\rangle = -\frac{2i\mu}{(2\pi)^{\frac{9}{2}}}(2\pi)^4\delta(k-p_2-p_1) \tag{3.49}$$

We would have got the same result if we had used the Feynman diagrams procedure.

The factor $\frac{1}{(2\pi)^{\frac{9}{2}}}$ is due to the convention we adopted for normalization of states (and therefore to the proportional constant that appear in the commutation relations) and for the $(2\pi)^{-\frac{1}{2}}$ in the Fourier anti-transform of the fields. If we had chosen in the normalization of states another convention such as:

$$\langle\vec{p}|\vec{q}\rangle = (2\pi)^{\frac{3}{2}}2E_p\delta(\vec{p}-\vec{q}) \tag{3.50}$$

it would have led to commutation relations like:

$$\left[\hat{a}(\vec{p_1}),\hat{a}^\dagger(\vec{p_2})\right] = (2\pi)^{\frac{3}{2}}\delta\left(\vec{p_1}-\vec{p_2}\right) \tag{3.51}$$

and in our computation, this kind of commutators appear exactly three times for each terms in $\langle\phi_\beta|\,\hat{S}^{(1)}\,|\phi_\alpha\rangle$, and so we would have got:

$$\langle\phi_\beta|\,\hat{S}^{(1)}\,|\phi_\alpha\rangle = -2i\mu(2\pi)^4\delta(k-p_2-p_1) \tag{3.52}$$

The same result we would have got if we had chosen the convention used by some book:

$$\hat{\Phi}(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2E_{\mathbf{k}}}}\left(\hat{a}(\vec{k})e^{-ikx}+\hat{a}^\dagger(\vec{k})e^{ikx}\right) \tag{3.53}$$

$$\langle\vec{p}|\vec{q}\rangle = (2\pi)^3 2E_p\delta(\vec{p}-\vec{q}) \tag{3.54}$$

It leads to:

$$\left[\hat{a}(\vec{p_1}), \hat{a}^\dagger(\vec{p_2})\right] = (2\pi)^3 \delta\left(\vec{p_1} - \vec{p_2}\right) \tag{3.55}$$

therefore, also with this convention, we would have obtained:

$$\langle\phi_\beta| \hat{S}^{(1)} |\phi_\alpha\rangle = -2i\mu(2\pi)^4 \delta(k - p_2 - p_1) \tag{3.56}$$

The value $\langle\phi_\beta| \hat{S}^{(1)} |\phi_\alpha\rangle$ (and so $\mathcal{M}_{\beta\alpha}$) depend by the used convention, but there is no problems if we are consistent with it. This means that we must pay attention, for istance, to the Feynman rules, or to the cross section and decay rate formulas we use and not mix up different conventions. Later we discuss the decay rate formula and how to deal with our result for $\langle\phi_\beta| \hat{S}^{(1)} |\phi_\alpha\rangle$.

The decay of η in two ξ particles is kinematically allowed if

$$m_\eta \geq 2m_\xi \tag{3.57}$$

Proof. Evaluating the invariant mass $M_{inv}^2 = E_{tot}^2 - |\vec{P}_{tot}|^2$, before the decay, in the center of mass frame:

$$M_{inv}^2 = m_\eta^2 \tag{3.58}$$

Evaluating it, after the decay, in the center of mass frame (in which $P_{tot} = 0$):

$$M_{inv}^2 = E_{tot}^2 = (E_1 + E_2)^2 = \left(\sqrt{m_\xi^2 + |\vec{p_1}|^2} + \sqrt{m_\xi^2 + |\vec{p_2}|^2}\right)^2 = \left(2\sqrt{m_\xi^2 + |\vec{p_1}|^2}\right)^2 \tag{3.59}$$

Hence:

$$m_\eta^2 = 4(m_\xi^2 + |\vec{p_1}|^2) \geq 4m_\xi^2 \tag{3.60}$$

■

The lifetime is defined as the inverse of the decay rate Γ. The generic formula for the differential decay rate for one particle η decaying into n ξ particles is very similar to the one of the differential cross section:

$$\mathrm{d}\Gamma = \frac{S}{2m_\eta} (\mathrm{d}\Pi_n) \left|\mathcal{M}\left(\eta \to \xi_1, \ldots, \xi_n\right)\right|^2 \tag{3.61}$$

where $(\mathrm{d}\Pi_n)$ is the invariant n -particle final state:

$$(\mathrm{d}\Pi_n) = \prod_{i=1,n} \frac{d^3p_i}{(2\pi)^3} \frac{1}{2E_i} (2\pi)^4 \delta_{(4)} \left(p_\eta - \sum_{i=1,n} p_i\right) \tag{3.62}$$

while S is the symmetry factor for identical particles in the final state:

$$S = \prod_{j=1,m} \frac{1}{n_j!} \tag{3.63}$$

where the index j runs over all m kinds of different particles in the final state and n_j is the number of identical particles j in the final state, such that: $\sum_{j=1,m} n_j = n$

In our case, we have two ξ particles in the final states, hence $S = \frac{1}{2}$. Therefore:

$$\mathrm{d}\Gamma = \frac{S}{2m_\eta}\left(\mathrm{d}\Pi_2\right)\left|\mathcal{M}\left(\eta \to \xi_1, \xi_2\right)\right|^2 = \tag{3.64}$$

$$= \frac{S}{2m_\eta}\frac{d^3p_1 d^3p_2}{(2\pi)^6}\frac{1}{2E_{p_1}2E_{p_2}}(2\pi)^4\delta_{(4)}\left(p_\eta - p_1 - p_2\right)\left|\mathcal{M}\left(\eta \to \xi_1, \xi_2\right)\right|^2 = \tag{3.65}$$

$$= \frac{1}{16m_\eta}\frac{d^3p_1 d^3p_2}{(2\pi)^6}\frac{(2\pi)^4}{E_{p_1}E_{p_2}}\delta_{(4)}\left(p_\eta - p_1 - p_2\right)\left|\mathcal{M}\left(\eta \to \xi_1, \xi_2\right)\right|^2 \tag{3.66}$$

The matrix element $\mathcal{M}\left(\eta \to \xi_1, \xi_2\right)$ that appears in this formula is not coherent with our convention. In order to use this formula we need to drop the factor $(2\pi)^{-\frac{9}{2}}$ which appear in our calculation (see what we discuss after eq 3.49).

Therefore, from eq 3.49 we have (at the first order in μ):

$$\mathcal{M}\left(\eta \to \xi_1, \xi_2\right) = -2\mu \tag{3.67}$$

So:

$$\mathrm{d}\Gamma = \frac{\mu^2}{4m_\eta(2\pi)^2}\frac{d^3p_1 d^3p_2}{E_{p_1}E_{p_2}}\delta_{(4)}\left(p_\eta - p_1 - p_2\right) \tag{3.68}$$

Considering that in the center of mass frame of η we have $E_\eta = m_\eta$ and $\vec{p}_\eta = \vec{0}$:

$$\Gamma = \int \mathrm{d}\Gamma = \frac{\mu^2}{4m_\eta(2\pi)^2}\int\frac{d^3p_1 d^3p_2}{E_{p_1}E_{p_2}}\delta_{(4)}\left(p_\eta - p_1 - p_2\right) = \tag{3.69}$$

$$= \frac{\mu^2}{4m_\eta(2\pi)^2}\int\frac{d^3p_1 d^3p_2}{E_{p_1}E_{p_2}}\delta\left(E_\eta - E_{p_1} - E_{p_2}\right)\delta_{(3)}\left(\vec{p}_\eta - \vec{p}_1 - \vec{p}_2\right) = \tag{3.70}$$

$$= \frac{\mu^2}{4m_\eta(2\pi)^2}\int\frac{d^3p_1 d^3p_2}{E_{p_1}E_{p_2}}\delta\left(m_\eta - E_{p_1} - E_{p_2}\right)\delta_{(3)}\left(-\vec{p}_1 - \vec{p}_2\right) = \tag{3.71}$$

$$= \frac{\mu^2}{4m_\eta(2\pi)^2}\int\frac{d^3p_1}{E_{p_1}^2}\delta\left(m_\eta - 2E_{p_1}\right) = \frac{\mu^2}{4m_\eta(2\pi)^2}\int\frac{d^3p_1}{E_{p_1}^2}\delta\left(m_\eta - 2\sqrt{m_\xi^2 + |\vec{p}_1|^2}\right) = \tag{3.72}$$

$$= \frac{\mu^2}{4\pi m_\eta}\int\frac{d|\vec{p}_1|}{m_\xi^2 + |\vec{p}_1|^2}|\vec{p}_1|^2\delta\left(m_\eta - 2\sqrt{m_\xi^2 + |\vec{p}_1|^2}\right) \tag{3.73}$$

We use the delta property:

$$\delta(f(x)) = \sum_{i=1}^{N}\frac{\delta\left(x - x_{0,i}\right)}{\left|f'\left(x_{0,i}\right)\right|} \tag{3.74}$$

where $\left\{x_{0,i}\right\}_{i=1,N}$ are the zeros of $f(x)$.

In our case :

$$f(|\vec{p}_1|) = m_\eta - 2\sqrt{m_\xi^2 + |\vec{p}_1|^2} \tag{3.75}$$

with zero:

$$|\vec{p}_{1,0}| = \sqrt{\frac{m_\eta^2}{4} - m_\xi^2} \tag{3.76}$$

Hence:

$$f'(|\vec{p}_{1,0}|) = -\frac{2\left|\vec{p}_{1,0}\right|}{\sqrt{m_\xi^2 + \left|\vec{p}_{1,0}\right|^2}} = -2\sqrt{1 - 4\frac{m_\xi^2}{m_\eta^2}} \tag{3.77}$$

We are sure that the quantity under square root is positive for condition 3.57. Therefore:

$$\Gamma = \frac{\mu^2}{4\pi m_\eta}\int \frac{d|\vec{p}_1|}{m_\xi^2+|\vec{p}_1|^2}|\vec{p}_1|^2\frac{\delta\left(|\vec{p}_1|-|\vec{p}_{1,0}|\right)}{2\sqrt{1-4\frac{m_\xi^2}{m_\eta^2}}} = \frac{\mu^2}{8\pi m_\eta}\frac{|\vec{p}_{1,0}|^2}{m_\xi^2+|\vec{p}_{1,0}|^2}\frac{1}{\sqrt{1-4\frac{m_\xi^2}{m_\eta^2}}} = \tag{3.78}$$

$$= \frac{\mu^2}{8\pi m_\eta}\sqrt{1-4\frac{m_\xi^2}{m_\eta^2}} \tag{3.79}$$

So, the lifetime is:

$$\tau = \frac{1}{\Gamma} = \frac{8\pi m_\eta}{\mu^2}\frac{1}{\sqrt{1-4\frac{m_\xi^2}{m_\eta^2}}} \tag{3.80}$$

The dimension of τ is M^{-1} as expected from eq.(3.6) .

4 Exercise 4: decay of a massive scalar particle

Compute the unpolarized, differential cross-section in the center of mass frame for the scattering of two positrons into two positrons at leading order in QED:

$$e^{+}+e^{+}\rightarrow e^{+}+e^{+} \tag{4.1}$$

- find the initial and final particle states, draw the relevant Feynman diagrams, and find their amplitudes [4pt];
- compute the unpolarized squared matrix element and express it in terms of the Mandelstam variables [4pt];
- calculate the differential cross section in the center of mass frame [3pt];
- discuss both the non-relativistic and the ultra-relativistic limits [2pt].

Solution

We can decompose our one flavour QED lagrangian into :

$$\hat{\mathcal{L}}=\hat{\mathcal{L}}_{0_{Dirac}}+\hat{\mathcal{L}}_{0_{EM}}+\hat{\mathcal{L}}_{\text{int}} \tag{4.2}$$

where:

$$\hat{\mathcal{L}}_{\text{int}}=-e\hat{A}_{\mu}\hat{\bar{\Psi}}\gamma^{\mu}\hat{\Psi} \tag{4.3}$$

We have:

$$\hat{H}=\hat{H}_0+\hat{V} \tag{4.4}$$

where $\hat{H}_0$ is the free hamiltonian while $\hat{V}(t)=\int d^3x\hat{\mathcal{U}}(x)$ is the interaction hamiltonian.

Since $\mathcal{L}_{\text{int}}$ does depend only on the fields and not on the derivatives of the fields, then :

$$\hat{\mathcal{L}}_{\text{int}}=-\hat{\mathcal{U}}=e\hat{A}_{\mu}\hat{\bar{\Psi}}\gamma^{\mu}\hat{\Psi} \tag{4.5}$$

We have the $\hat{S}$ matrix:

$$\hat{S}=\hat{U}_I(+\infty,-\infty) \tag{4.6}$$

where:

$$\hat{U}_I(+\infty,-\infty)=\lim_{\substack{\tau\to+\infty\\ \tau_0\to-\infty}}\exp\left(iH_0\tau\right)\exp\left(-iH\left(\tau-\tau_0\right)\right)\exp\left(-iH_0\tau_0\right) \tag{4.7}$$

We can show that:

$$\hat{S}=\hat{U}_I(+\infty,-\infty)=\mathbb{1}+\sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{\infty}d^4x_1\cdots d^4x_nT\left\{\hat{\mathcal{U}}_I\left(x_1\right)\cdots\hat{\mathcal{U}}_I\left(x_n\right)\right\}= \tag{4.8}$$

$$=T\left(\exp\left(-i\int_{-\infty}^{+\infty}d^4x\hat{\mathcal{U}}_I(x)\right)\right)=\sum_{n=0}^{\infty}\hat{S}^{(n)} \tag{4.9}$$

where:

$$\hat{\mathcal{U}}_I(t,\vec{x})=\hat{U}_0^{\dagger}(t)\mathcal{U}_s(\vec{x})\hat{U}_0(t) \tag{4.10}$$

where $\mathcal{U}_s(\vec{x})$ is the interaction potential operator in the Schrodinger picture and:

$$\hat{U}_0(t)=exp(-i\hat{H}_0t) \tag{4.11}$$

In our case:

$$\hat{\mathcal{U}}_I(x)=e\hat{A}_{I\mu}\hat{\bar{\Psi}}_I\gamma^{\mu}\hat{\Psi}_I \tag{4.12}$$

Proof.

$$\hat{\mathcal{U}}_I(x) = \hat{U}_0^\dagger(t)\mathcal{U}_s(\vec{x})\hat{U}_0(t) = e\hat{U}_0^\dagger(t)\hat{A}_{\mu_s}(x)\hat{\bar{\Psi}}_s(x)\gamma^\mu\hat{\Psi}_s(x)\hat{U}_0(t) = \tag{4.13}$$

$$= e\hat{U}_0^\dagger(t)\hat{A}_{\mu_s}(x)\hat{U}_0(t)\hat{U}_0^\dagger(t)\hat{\bar{\Psi}}_s(x)\hat{U}_0(t)\hat{U}_0^\dagger(t)\gamma^\mu\hat{\Psi}_s(x)\hat{U}_0(t) = \tag{4.14}$$

$$= e\hat{A}_{I\mu}(x)\hat{\bar{\Psi}}_I(x)\gamma^\mu\hat{\Psi}_I(x) \tag{4.15}$$

■

Since the fields in the interaction picture of an interacting theory are identical to the fields in the Heisenberg picture of the corresponding free theory, we have:

$$\hat{A}_I^\nu(x) = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2E_{\mathbf{k}}}}\sum_{\lambda=0}^{3}\left(\hat{a}(\vec{k},\lambda)\epsilon^\nu(\vec{k},\lambda)e^{-ikx} + \hat{a}^\dagger(\vec{k},\lambda)\epsilon^{\nu*}(\vec{k},\lambda)e^{ikx}\right) \tag{4.16}$$

$$\hat{\Psi}_I(x) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}\sqrt{2E_{\mathbf{p}}}}\sum_{s=1,2}\left(\hat{c}(\vec{p},s)u(\vec{p},s)e^{-ipx} + \hat{d}^\dagger(\vec{p},s)v(\vec{p},s)e^{ipx}\right) \tag{4.17}$$

$$\hat{\bar{\Psi}}_I(x) = \hat{\Psi}_I^\dagger(x)\gamma^0 = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}\sqrt{2E_{\mathbf{p}}}}\sum_{s=1,2}\left(\hat{d}(\vec{p},s)\bar{v}(\vec{p},s)e^{-ipx} + \hat{c}^\dagger(\vec{p},s)\bar{u}(\vec{p},s)e^{ipx}\right) \tag{4.18}$$

The initial and final states are given by:

$$|\phi_\alpha\rangle = \left|e_1^+ : \vec{k_1}, r\,; e_2^+ : \vec{k}_2, s\right\rangle = \sqrt{2E_{k_1}2E_{k_2}}\hat{d}^\dagger(\vec{k}_1, r)\hat{d}^\dagger(\vec{k}_2, s)\,|0\rangle \tag{4.19}$$

$$|\phi_\beta\rangle = \left|e_1^+ : \vec{p_1}, t\,; e_2^+ : \vec{p}_2, q\right\rangle = \sqrt{2E_{p_1}2E_{p_2}}\hat{d}^\dagger(\vec{p}_1, t)\hat{d}^\dagger(\vec{p}_2, q)\,|0\rangle \tag{4.20}$$

We have that:

$$\langle\phi_\beta|\,\hat{S}^{(1)}\,|\phi_\alpha\rangle = 0 \tag{4.21}$$

since we have no photons in initial and final states and so the operators $\hat{a}$ or $\hat{a}^\dagger$, which appear inside $\hat{A}_I$ in the interaction potential, will get 0 acting on the vacuum.

Hence, we need to go to second order for a propagator to appear.

$$\langle\phi_\beta|\,\hat{S}^{(2)}\,|\phi_\alpha\rangle = \frac{(ie)^2}{2}\sqrt{2E_{p_1}2E_{p_2}2E_{k_1}2E_{k_2}}\int_{-\infty}^{+\infty} d^4x\,d^4y \tag{4.22}$$

$$\langle 0|\,\hat{d}(\vec{p}_1, t)\hat{d}(\vec{p}_2, q)T\left(\left(\hat{A}_{I\mu}\hat{\bar{\Psi}}_I\gamma^\mu\hat{\Psi}_I\right)_x\left(\hat{A}_{I\mu}\hat{\bar{\Psi}}_I\gamma^\mu\hat{\Psi}_I\right)_y\right)\hat{d}^\dagger(\vec{k}_1, r)\hat{d}^\dagger(\vec{k}_2, s)\,|0\rangle \tag{4.23}$$

$$\tag{4.24}$$

Taking into account that we have only positrons in initial and final states and that all relevant Feynman diagrams are connected graphes, among the various terms that appear in eq.(4.24) applying Wick's theorem, the only one we must consider is:

$$\langle\phi_\beta|\,\hat{S}^{(2)}\,|\phi_\alpha\rangle = \frac{(ie)^2}{2}\sqrt{2E_{p_1}2E_{p_2}2E_{k_1}2E_{k_2}}\int_{-\infty}^{+\infty} d^4x\,d^4y \tag{4.25}$$

$$\langle 0|\,\hat{d}(\vec{p}_1, t)\hat{d}(\vec{p}_2, q) : \left(\overbrace{A_{I\mu_x}\hat{\bar{\Psi}}_{I_x}\gamma^\mu\hat{\Psi}_{I_x}}A_{I\mu_y}\hat{\bar{\Psi}}_{I_y}\gamma^\mu\hat{\Psi}_{I_y}\right) : \hat{d}^\dagger(\vec{k}_1, r)\hat{d}^\dagger(\vec{k}_2, s)\,|0\rangle \tag{4.26}$$

- $\hat{d}(\vec{p}_1, t)$ in $|\phi_\beta\rangle$ must contract with $\hat{d}^\dagger$ in $\hat{\Psi}(x)$ or $\hat{\Psi}(y)$
- $\hat{d}^\dagger(\vec{k}_1, r)$ in $|\phi_\alpha\rangle$ must contract with $\hat{d}$ in $\hat{\bar{\Psi}}(x)$ or $\hat{\bar{\Psi}}(y)$
  Once we make these choices, we fix the others:
- $\hat{d}(\vec{p}_2, q)$ in $|\phi_\beta\rangle$ must contract with $\hat{d}^\dagger$ in $\hat{\Psi}(x)$ or $\hat{\Psi}(y)$
- $\hat{d}^\dagger(\vec{k}_2, s)$ in $|\phi_\alpha\rangle$ must contract with $\hat{d}$ in $\hat{\bar{\Psi}}(x)$ or $\hat{\bar{\Psi}}(y)$

So we have the following possibilities:

1. $\hat{d}(\vec{p}_1, t)$ in $|\phi_\beta\rangle$ contracted with $\hat{d}^\dagger$ in $\hat{\Psi}(x)$, $\hat{d}^\dagger(\vec{k}_1, r)$ in $|\phi_\alpha\rangle$ with $\hat{d}$ in $\hat{\bar{\Psi}}(x)$, $\hat{d}(\vec{p}_2, q)$ in $|\phi_\beta\rangle$ with $\hat{d}^\dagger$ in $\hat{\Psi}(y)$, $\hat{d}^\dagger(\vec{k}_2, s)$ in $|\phi_\alpha\rangle$ with $\hat{d}$ in $\hat{\bar{\Psi}}(y)$.
2. $\hat{d}(\vec{p}_1, t)$ in $|\phi_\beta\rangle$ contracted with $\hat{d}^\dagger$ in $\hat{\Psi}(x)$, $\hat{d}^\dagger(\vec{k}_1, r)$ in $|\phi_\alpha\rangle$ with $\hat{d}$ in $\hat{\bar{\Psi}}(y)$, $\hat{d}(\vec{p}_2, q)$ in $|\phi_\beta\rangle$ with $\hat{d}^\dagger$ in $\hat{\Psi}(y)$, $\hat{d}^\dagger(\vec{k}_2, s)$ in $|\phi_\alpha\rangle$ with $\hat{d}$ in $\hat{\bar{\Psi}}(x)$.
3. $\hat{d}(\vec{p}_1, t)$ in $|\phi_\beta\rangle$ contracted with $\hat{d}^\dagger$ in $\hat{\Psi}(y)$, $\hat{d}^\dagger(\vec{k}_1, r)$ in $|\phi_\alpha\rangle$ with $\hat{d}$ in $\hat{\bar{\Psi}}(x)$, $\hat{d}(\vec{p}_2, q)$ in $|\phi_\beta\rangle$ with $\hat{d}^\dagger$ in $\hat{\Psi}(x)$, $\hat{d}^\dagger(\vec{k}_2, s)$ in $|\phi_\alpha\rangle$ with $\hat{d}$ in $\hat{\bar{\Psi}}(y)$.
4. $\hat{d}(\vec{p}_1, t)$ in $|\phi_\beta\rangle$ contracted with $\hat{d}^\dagger$ in $\hat{\Psi}(y)$, $\hat{d}^\dagger(\vec{k}_1, r)$ in $|\phi_\alpha\rangle$ with $\hat{d}$ in $\hat{\bar{\Psi}}(y)$, $\hat{d}(\vec{p}_2, q)$ in $|\phi_\beta\rangle$ with $\hat{d}^\dagger$ in $\hat{\Psi}(x)$, $\hat{d}^\dagger(\vec{k}_2, s)$ in $|\phi_\alpha\rangle$ with $\hat{d}$ in $\hat{\bar{\Psi}}(x)$.

Since x and y are integration variables, Feynman diagrams 1) and 4) (we define these two as "Case A") are the same as well as 2) and 3) (we define these two as "Case B"). This fact will imply a factor 2 that cancel the 2 which appears in the denominator of 4.26.

The Feynman diagram associated to 1) and 4) is:

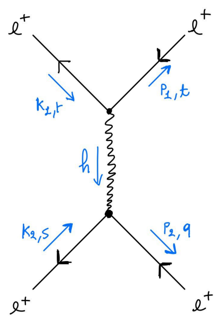

Figure 1: Feynman diagram associated to cases 1) and 4)

Its associated amplitude is :

$$i\mathcal{M}^t_{\beta\alpha} = \bar{v}(\vec{k}_1, r)(-ie\gamma^\mu)v(\vec{p}_1, t)\frac{-i\eta_{\mu\nu}}{(p_1 - k_1)^2 + i\epsilon}\bar{v}(\vec{k}_2, s)(-ie\gamma^\nu)v(\vec{p}_2, q) \tag{4.27}$$

where we have used the fact due to momentum conservation that :

$$h = k_1 - p_1 \tag{4.28}$$

The Feynman diagram associated to 2) and 3) is:

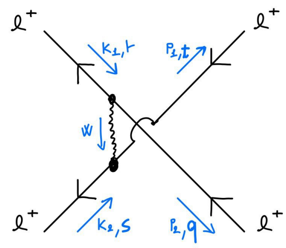

Figure 2: Feynman diagram associated to cases 2) and 3)

Its associated amplitude is :

$$i\mathcal{M}^u_{\beta\alpha} = \bar{v}(\vec{k}_1, r)(-ie\gamma^\mu)v(\vec{p}_2, q)\frac{-i\eta_{\mu\nu}}{(p_2 - k_1)^2 + i\epsilon}\bar{v}(\vec{k}_2, s)(-ie\gamma^\nu)v(\vec{p}_1, t) \tag{4.29}$$

where we have used the fact due to momentum conservation that :

$$w = k_1 - p_2 \tag{4.30}$$

(The subscript "t" and "u" on the amplitudes are due to Mandelstam notation that we will discuss later)

Now before to sum $i\mathcal{M}^t_{\beta\alpha}$ and $i\mathcal{M}^u_{\beta\alpha}$, we must pay attention to relative sign between them.

It is possible to understand the relative sign by brute force calculations considering that for each time we commute two fermionic operator we must take into account a minus sign.

We have:

$$\langle\phi_\beta|\,\hat{S}^{(2)}\,|\phi_\alpha\rangle = \frac{(ie)^2}{2}\sqrt{2E_{p_1}2E_{p_2}2E_{k_1}2E_{k_2}}\int_{-\infty}^{+\infty} d^4x\, d^4y \tag{4.31}$$

$$\langle 0|\,\hat{d}(\vec{p}_1, t)\hat{d}(\vec{p}_2, q) : \left(\overline{A_{I\mu_x}\hat{\bar{\Psi}}_{I_x}\gamma^\mu\hat{\Psi}_{I_x}A}_{I\mu_y}\hat{\bar{\Psi}}_{I_y}\gamma^\mu\hat{\Psi}_{I_y}\right) : \hat{d}^\dagger(\vec{k}_1, r)\hat{d}^\dagger(\vec{k}_2, s)\,|0\rangle \tag{4.32}$$

For $i\mathcal{M}^t_{\beta\alpha}$ we had: $\hat{d}(\vec{p}_1, t)$ in $|\phi_\beta\rangle$ contracted with $\hat{d}^\dagger$ in $\hat{\Psi}(x)$, $\hat{d}^\dagger(\vec{k}_1, r)$ in $|\phi_\alpha\rangle$ with $\hat{d}$ in $\hat{\bar{\Psi}}(x)$, $\hat{d}(\vec{p}_2, q)$ in $|\phi_\beta\rangle$ with $\hat{d}^\dagger$ in $\hat{\Psi}(y)$, $\hat{d}^\dagger(\vec{k}_2, s)$ in $|\phi_\alpha\rangle$ with $\hat{d}$ in $\hat{\bar{\Psi}}(y)$. Hence writing only the operators (in the following step we put Ψ to indicate where the operators come from):

$$\langle 0|\,\hat{d}(\vec{p}_1, t)\hat{d}(\vec{p}_2, q) : (\hat{d})_{\hat{\bar{\Psi}}(x)}(\hat{d}^\dagger)_{\hat{\Psi}(x)}(\hat{d})_{\hat{\bar{\Psi}}(y)}(\hat{d}^\dagger)_{\hat{\Psi}(y)} : \hat{d}^\dagger(\vec{k}_1, r)\hat{d}^\dagger(\vec{k}_2, s)\,|0\rangle = \tag{4.33}$$

$$= (-1)^{1+2}\langle 0|\,\hat{d}(\vec{p}_1, t)\hat{d}(\vec{p}_2, q) : (\hat{d}^\dagger)_{\hat{\Psi}(x)}(\hat{d}^\dagger)_{\hat{\Psi}(y)}(\hat{d})_{\hat{\bar{\Psi}}(x)}(\hat{d})_{\hat{\bar{\Psi}}(y)} : \hat{d}^\dagger(\vec{k}_1, r)\hat{d}^\dagger(\vec{k}_2, s)\,|0\rangle \tag{4.34}$$

So, we want to put:

- $\hat{d}(\vec{p}_1, t)$ near $(\hat{d}^\dagger)_{\hat{\Psi}(x)} \longrightarrow (-1)^1$
- $\hat{d}^\dagger(\vec{k}_1, r)$ near $(\hat{d})_{\hat{\bar{\Psi}}(x)} \longrightarrow (-1)^1$
- $\hat{d}(\vec{p}_2, q)$ near $(\hat{d}^\dagger)_{\hat{\Psi}(y)} \longrightarrow (-1)^0$

- $\hat{d}^\dagger(\vec{k}_2, s)$ near $(\hat{d})_{\hat{\bar{\Psi}}(y)} \longrightarrow (-1)^0$

So we have in total a factor $(-1)^{1+2+1+1+0+0} = -1$

For $i\mathcal{M}^u_{\beta\alpha}$ we had: $\hat{d}(\vec{p}_1, t)$ in $|\phi_\beta\rangle$ contracted with $\hat{d}^\dagger$ in $\hat{\Psi}(x)$, $\hat{d}^\dagger(\vec{k}_1, r)$ in $|\phi_\alpha\rangle$ with $\hat{d}$ in $\hat{\bar{\Psi}}(y)$, $\hat{d}(\vec{p}_2, q)$ in $|\phi_\beta\rangle$ with $\hat{d}^\dagger$ in $\hat{\Psi}(y)$, $\hat{d}^\dagger(\vec{k}_2, s)$ in $|\phi_\alpha\rangle$ with $\hat{d}$ in $\hat{\bar{\Psi}}(x)$. Hence writing only the operators:

$$\langle 0| \hat{d}(\vec{p}_1, t)\hat{d}(\vec{p}_2, q) : (\hat{d})_{\hat{\bar{\Psi}}(x)}(\hat{d}^\dagger)_{\hat{\Psi}(x)}(\hat{d})_{\hat{\bar{\Psi}}(x)}(\hat{d}^\dagger)_{\hat{\Psi}(y)} : \hat{d}^\dagger(\vec{k}_1, r)\hat{d}^\dagger(\vec{k}_2, s) |0\rangle = \tag{4.35}$$

$$= (-1)^{1+2} \langle 0| \hat{d}(\vec{p}_1, t)\hat{d}(\vec{p}_2, q) : (\hat{d}^\dagger)_{\hat{\Psi}(x)}(\hat{d}^\dagger)_{\hat{\Psi}(y)}(\hat{d})_{\hat{\bar{\Psi}}(x)}(\hat{d})_{\hat{\bar{\Psi}}(y)} : \hat{d}^\dagger(\vec{k}_1, r)\hat{d}^\dagger(\vec{k}_2, s) |0\rangle \tag{4.36}$$

So, we want to put:

- $\hat{d}(\vec{p}_1, t)$ near $(\hat{d}^\dagger)_{\hat{\Psi}(x)} \longrightarrow (-1)^1$
- $\hat{d}^\dagger(\vec{k}_1, r)$ near $(\hat{d})_{\hat{\bar{\Psi}}(y)} \longrightarrow (-1)^0$
- $\hat{d}(\vec{p}_2, q)$ near $(\hat{d}^\dagger)_{\hat{\Psi}(y)} \longrightarrow (-1)^0$
- $\hat{d}^\dagger(\vec{k}_2, s)$ near $(\hat{d})_{\hat{\bar{\Psi}}(x)} \longrightarrow (-1)^0$

So we have in total a factor $(-1)^{1+2+1+0+0+0} = 1$

Therefore, the relative sign is -1.

We have:

$$i\mathcal{M}_{\alpha\beta} = i\mathcal{M}^u_{\alpha\beta} - i\mathcal{M}^t_{\alpha\beta} \tag{4.37}$$

(we could have put also the - sign in front of the first and the + in front of the second)

$$|\mathcal{M}_{\alpha\beta}|^2 = |i\mathcal{M}_{\alpha\beta}|^2 = (i\mathcal{M}_{\alpha\beta})^*(i\mathcal{M}_{\alpha\beta}) \tag{4.38}$$

$$|\mathcal{M}(r, s \to t, q)|^2 \equiv |\mathcal{M}_{\alpha\beta}|^2 \tag{4.39}$$

In order to compute the unpolarized squared matrix element we have to sum over all possible final states and average over all possible initial states:

$$\mathcal{M}_{\text{unpol}} = \frac{1}{2 \cdot 2} \sum_{t,q} \sum_{r,s} |\mathcal{M}(r, s \to t, q)|^2 \tag{4.40}$$

$$i\mathcal{M}(r, s \to t, q) = i\mathcal{M}^u_{\alpha\beta} - i\mathcal{M}^t_{\alpha\beta} = \tag{4.41}$$

$$= \bar{v}(\vec{k}_1, r)(-ie\gamma^\mu)v(\vec{p}_2, q)\frac{-i\eta_{\mu\nu}}{(p_2 - k_1)^2 + i\epsilon}\bar{v}(\vec{k}_2, s)(-ie\gamma^\nu)v(\vec{p}_1, t) \tag{4.42}$$

$$- \bar{v}(\vec{k}_1, r)(-ie\gamma^\mu)v(\vec{p}_1, t)\frac{-i\eta_{\mu\nu}}{(p_1 - k_1)^2 + i\epsilon}\bar{v}(\vec{k}_2, s)(-ie\gamma^\nu)v(\vec{p}_2, q) \tag{4.43}$$

$$= ie^2[\bar{v}(\vec{k}_1, r)(\gamma^\mu)v(\vec{p}_2, q)\frac{\eta_{\mu\nu}}{(p_2 - k_1)^2 + i\epsilon}\bar{v}(\vec{k}_2, s)(\gamma^\nu)v(\vec{p}_1, t) \tag{4.44}$$

$$- \bar{v}(\vec{k}_1, r)(\gamma^\mu)v(\vec{p}_1, t)\frac{\eta_{\mu\nu}}{(p_1 - k_1)^2 + i\epsilon}\bar{v}(\vec{k}_2, s)(\gamma^\nu)v(\vec{p}_2, q)] \tag{4.45}$$

$$= ie^2[\bar{v}_\alpha(\vec{k}_1, r)\gamma^\mu_{\alpha\beta}v_\beta(\vec{p}_2, q)\frac{\eta_{\mu\nu}}{(p_2 - k_1)^2 + i\epsilon}\bar{v}_\delta(\vec{k}_2, s)\gamma^\nu_{\delta\zeta}v_\zeta(\vec{p}_1, t) \tag{4.46}$$

$$- \bar{v}_\alpha(\vec{k}_1, r)\gamma^\mu_{\alpha\beta}v_\beta(\vec{p}_1, t)\frac{\eta_{\mu\nu}}{(p_1 - k_1)^2 + i\epsilon}\bar{v}_\delta(\vec{k}_2, s)\gamma^\nu_{\delta\zeta}v_\zeta(\vec{p}_2, q)] \tag{4.47}$$

We have:

$$(i\mathcal{M}_{\alpha\beta})^* = -ie^2[\bar{v}(\vec{p}_2,q)\gamma_\nu v(\vec{k}_1,r)\frac{1}{w^2}\bar{v}(\vec{p}_1,t)\gamma^\nu v(\vec{k}_2,s) - \bar{v}(\vec{p}_1,t)\gamma_\nu v(\vec{k}_1,r)\frac{1}{h^2}\bar{v}(\vec{p}_2,q)\gamma^\nu v(\vec{k}_2,s)] \tag{4.48}$$

Proof. In the following steps, we are going to use:

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \ , \ (A^*)_{ij} = (A^\dagger)_{ji} \ , \ \gamma^0\gamma^0 = \mathbb{1} \ , \ \bar{v}_\alpha = v^*_\beta\gamma^0_{\beta\alpha} \tag{4.49}$$

$$\begin{aligned}
(i\mathcal{M}_{\alpha\beta})^* = &-ie^2[\bar{v}_\alpha(\vec{k}_1,r)\gamma^\mu_{\alpha\beta}v_\beta(\vec{p}_2,q)\frac{\eta_{\mu\nu}}{(p_2-k_1)^2+i\epsilon}\bar{v}_\delta(\vec{k}_2,s)\gamma^\nu_{\delta\zeta}v_\zeta(\vec{p}_1,t)\\
&-\bar{v}_\alpha(\vec{k}_1,r)\gamma^\mu_{\alpha\beta}v_\beta(\vec{p}_1,t)\frac{\eta_{\mu\nu}}{(p_1-k_1)^2+i\epsilon}\bar{v}_\delta(\vec{k}_2,s)\gamma^\nu_{\delta\zeta}v_\zeta(\vec{p}_2,q)]^* =\\
=&-ie^2[\bar{v}^*_\alpha(\vec{k}_1,r)\gamma^{\mu*}_{\alpha\beta}v^*_\beta(\vec{p}_2,q)\frac{\eta_{\mu\nu}}{(p_2-k_1)^2-i\epsilon}\bar{v}^*_\delta(\vec{k}_2,s)\gamma^{\nu*}_{\delta\zeta}v^*_\zeta(\vec{p}_1,t)\\
&-\bar{v}^*_\alpha(\vec{k}_1,r)\gamma^{\mu*}_{\alpha\beta}v^*_\beta(\vec{p}_1,t)\frac{\eta_{\mu\nu}}{(p_1-k_1)^2-i\epsilon}\bar{v}^*_\delta(\vec{k}_2,s)\gamma^{\nu*}_{\delta\zeta}v^*_\zeta(\vec{p}_2,q)] =\\
=&-ie^2[v_\iota(\vec{k}_1,r)\gamma^{0*}_{\iota\alpha}(\gamma^{\mu\dagger})_{\beta\alpha}v^*_\beta(\vec{p}_2,q)\frac{\eta_{\mu\nu}}{(p_2-k_1)^2-i\epsilon}v_\kappa(\vec{k}_2,s)\gamma^{0*}_{\kappa\delta}(\gamma^{\nu\dagger})_{\zeta\delta}v^*_\zeta(\vec{p}_1,t)\\
&-v_\iota(\vec{k}_1,r)\gamma^{0*}_{\iota\alpha}(\gamma^{\mu\dagger})_{\beta\alpha}v^*_\beta(\vec{p}_1,t)\frac{\eta_{\mu\nu}}{(p_1-k_1)^2-i\epsilon}v_\kappa(\vec{k}_2,s)\gamma^{0*}_{\kappa\delta}(\gamma^{\nu\dagger})_{\zeta\delta}v^*_\zeta(\vec{p}_2,q)] =\\
=&-ie^2[v_\iota(\vec{k}_1,r)\gamma^{0*}_{\iota\alpha}(\gamma^0\gamma^\mu\gamma^0)_{\beta\alpha}v^*_\beta(\vec{p}_2,q)\frac{\eta_{\mu\nu}}{(p_2-k_1)^2-i\epsilon}v_\kappa(\vec{k}_2,s)\gamma^{0*}_{\kappa\delta}(\gamma^0\gamma^\nu\gamma^0)_{\zeta\delta}v^*_\zeta(\vec{p}_1,t)\\
&-v_\iota(\vec{k}_1,r)\gamma^{0*}_{\iota\alpha}(\gamma^0\gamma^\mu\gamma^0)_{\beta\alpha}v^*_\beta(\vec{p}_1,t)\frac{\eta_{\mu\nu}}{(p_1-k_1)^2-i\epsilon}v_\kappa(\vec{k}_2,s)\gamma^{0*}_{\kappa\delta}(\gamma^0\gamma^\nu\gamma^0)_{\zeta\delta}v^*_\zeta(\vec{p}_2,q)] =\\
=&-ie^2[v_\iota(\vec{k}_1,r)\gamma^0_{\alpha\iota}\gamma^0_{\beta\tau}\gamma^\mu_{\tau\xi}\gamma^0_{\xi\alpha}v^*_\beta(\vec{p}_2,q)\frac{\eta_{\mu\nu}}{(p_2-k_1)^2-i\epsilon}v_\kappa(\vec{k}_2,s)\gamma^0_{\delta\kappa}\gamma^0_{\zeta\epsilon}\gamma^\nu_{\epsilon\eta}\gamma^0_{\eta\delta}v^*_\zeta(\vec{p}_1,t)\\
&-v_\iota(\vec{k}_1,r)\gamma^0_{\alpha\iota}\gamma^0_{\beta\tau}\gamma^\mu_{\tau\xi}\gamma^0_{\xi\alpha}v^*_\beta(\vec{p}_1,t)\frac{\eta_{\mu\nu}}{(p_1-k_1)^2-i\epsilon}v_\kappa(\vec{k}_2,s)\gamma^0_{\delta\kappa}\gamma^0_{\zeta\epsilon}\gamma^\nu_{\epsilon\eta}\gamma^0_{\eta\delta}v^*_\zeta(\vec{p}_2,q)] =\\
=&-ie^2[v^*_\beta(\vec{p}_2,q)\gamma^0_{\beta\tau}\gamma^\mu_{\tau\xi}\gamma^0_{\xi\alpha}\gamma^0_{\alpha\iota}v_\iota(\vec{k}_1,r)\frac{\eta_{\mu\nu}}{(p_2-k_1)^2-i\epsilon}v^*_\zeta(\vec{p}_1,t)\gamma^0_{\zeta\epsilon}\gamma^\nu_{\epsilon\eta}\gamma^0_{\eta\delta}\gamma^0_{\delta\kappa}v_\kappa(\vec{k}_2,s)\\
&-v^*_\beta(\vec{p}_1,t)\gamma^0_{\beta\tau}\gamma^\mu_{\tau\xi}\gamma^0_{\xi\alpha}\gamma^0_{\alpha\iota}v_\iota(\vec{k}_1,r)\frac{\eta_{\mu\nu}}{(p_1-k_1)^2-i\epsilon}v^*_\zeta(\vec{p}_2,q)\gamma^0_{\zeta\epsilon}\gamma^\nu_{\epsilon\eta}\gamma^0_{\eta\delta}\gamma^0_{\delta\kappa}v_\kappa(\vec{k}_2,s)] =\\
=&-ie^2[\bar{v}(\vec{p}_2,q)(\gamma^\mu)v(\vec{k}_1,r)\frac{\eta_{\mu\nu}}{(p_2-k_1)^2-i\epsilon}\bar{v}(\vec{p}_1,t)(\gamma^\nu)v(\vec{k}_2,s)\\
&-\bar{v}(\vec{p}_1,t)(\gamma^\mu)v(\vec{k}_1,r)\frac{\eta_{\mu\nu}}{(p_1-k_1)^2-i\epsilon}\bar{v}(\vec{p}_2,q)(\gamma^\nu)v(\vec{k}_2,s)] =\\
=&ie^2[\bar{v}(\vec{p}_2,q)\gamma_\nu v(\vec{k}_1,r)\frac{1}{w^2}\bar{v}(\vec{p}_1,t)\gamma^\nu v(\vec{k}_2,s)\\
&-\bar{v}(\vec{p}_1,t)\gamma_\nu v(\vec{k}_1,r)\frac{1}{h^2}\bar{v}(\vec{p}_2,q)\gamma^\nu v(\vec{k}_2,s)]
\end{aligned}$$

■

$$|\mathcal{M}(r,s\to t,q)|^2 = (i\mathcal{M}_{\alpha\beta})^*(i\mathcal{M}_{\alpha\beta}) = \tag{4.50}$$

$$= e^4[\bar{v}(\vec{p}_2,q)\gamma_\nu v(\vec{k}_1,r)\frac{1}{w^2}\bar{v}(\vec{p}_1,t)\gamma^\nu v(\vec{k}_2,s) - \bar{v}(\vec{p}_1,t)\gamma_\nu v(\vec{k}_1,r)\frac{1}{h^2}\bar{v}(\vec{p}_2,q)\gamma^\nu v(\vec{k}_2,s)]\cdot \tag{4.51}$$

$$\cdot\, [\bar{v}(\vec{k}_1,r)\gamma_\mu v(\vec{p}_2,q)\frac{1}{w^2}\bar{v}(\vec{k}_2,s)\gamma^\mu v(\vec{p}_1,t) - \bar{v}(\vec{k}_1,r)\gamma_\mu v(\vec{p}_1,t)\frac{1}{h^2}\bar{v}(\vec{k}_2,s)\gamma^\mu v(\vec{p}_2,q)] \tag{4.52}$$

$$= e^4[\bar{v}(\vec{p}_2,q)\gamma_\nu v(\vec{k}_1,r)\frac{1}{w^2}\bar{v}(\vec{p}_1,t)\gamma^\nu v(\vec{k}_2,s)\cdot\bar{v}(\vec{k}_1,r)\gamma_\mu v(\vec{p}_2,q)\frac{1}{w^2}\bar{v}(\vec{k}_2,s)\gamma^\mu v(\vec{p}_1,t) \tag{4.53}$$

$$-\,\bar{v}(\vec{p}_2,q)\gamma_\nu v(\vec{k}_1,r)\frac{1}{w^2}\bar{v}(\vec{p}_1,t)\gamma^\nu v(\vec{k}_2,s)\cdot\bar{v}(\vec{k}_1,r)\gamma_\mu v(\vec{p}_1,t)\frac{1}{h^2}\bar{v}(\vec{k}_2,s)\gamma^\mu v(\vec{p}_2,q) \tag{4.54}$$

$$-\,\bar{v}(\vec{p}_1,t)\gamma_\nu v(\vec{k}_1,r)\frac{1}{h^2}\bar{v}(\vec{p}_2,q)\gamma^\nu v(\vec{k}_2,s)\cdot\bar{v}(\vec{k}_1,r)\gamma_\mu v(\vec{p}_2,q)\frac{1}{w^2}\bar{v}(\vec{k}_2,s)\gamma^\mu v(\vec{p}_1,t) \tag{4.55}$$

$$+\,\bar{v}(\vec{p}_1,t)\gamma_\nu v(\vec{k}_1,r)\frac{1}{h^2}\bar{v}(\vec{p}_2,q)\gamma^\nu v(\vec{k}_2,s)\cdot\bar{v}(\vec{k}_1,r)\gamma_\mu v(\vec{p}_1,t)\frac{1}{h^2}\bar{v}(\vec{k}_2,s)\gamma^\mu v(\vec{p}_2,q)] \tag{4.56}$$

$$\tag{4.57}$$

Hence, we have:

$$|\mathcal{M}_{\text{unpol}}|^2 = \frac{1}{4}\sum_{t,q}\sum_{r,s}|\mathcal{M}(r,s\to t,q)|^2 = \tag{4.58}$$

$$= \frac{e^4}{4}\sum_{t,q}\sum_{r,s}[\bar{v}(\vec{p}_2,q)\gamma_\nu v(\vec{k}_1,r)\frac{1}{w^2}\bar{v}(\vec{p}_1,t)\gamma^\nu v(\vec{k}_2,s)\cdot\bar{v}(\vec{k}_1,r)\gamma_\mu v(\vec{p}_2,q)\frac{1}{w^2}\bar{v}(\vec{k}_2,s)\gamma^\mu v(\vec{p}_1,t) \tag{4.59}$$

$$-\,\bar{v}(\vec{p}_2,q)\gamma_\nu v(\vec{k}_1,r)\frac{1}{w^2}\bar{v}(\vec{p}_1,t)\gamma^\nu v(\vec{k}_2,s)\cdot\bar{v}(\vec{k}_1,r)\gamma_\mu v(\vec{p}_1,t)\frac{1}{h^2}\bar{v}(\vec{k}_2,s)\gamma^\mu v(\vec{p}_2,q) \tag{4.60}$$

$$-\,\bar{v}(\vec{p}_1,t)\gamma_\nu v(\vec{k}_1,r)\frac{1}{h^2}\bar{v}(\vec{p}_2,q)\gamma^\nu v(\vec{k}_2,s)\cdot\bar{v}(\vec{k}_1,r)\gamma_\mu v(\vec{p}_2,q)\frac{1}{w^2}\bar{v}(\vec{k}_2,s)\gamma^\mu v(\vec{p}_1,t) \tag{4.61}$$

$$+\,\bar{v}(\vec{p}_1,t)\gamma_\nu v(\vec{k}_1,r)\frac{1}{h^2}\bar{v}(\vec{p}_2,q)\gamma^\nu v(\vec{k}_2,s)\cdot\bar{v}(\vec{k}_1,r)\gamma_\mu v(\vec{p}_1,t)\frac{1}{h^2}\bar{v}(\vec{k}_2,s)\gamma^\mu v(\vec{p}_2,q)] \tag{4.62}$$

$$\tag{4.63}$$

In order to use completeness relation:

$$\sum_{s=1}^{2} v_i(\vec{p},s)\bar{v}_j(\vec{p},s) = (p^\beta\gamma_\beta - m)_{ij} \tag{4.64}$$

I rewrite the previous equations with indexes:

$$|\mathcal{M}_{\text{unpol}}|^2 = \frac{e^4}{4}\sum_{t,q}\sum_{r,s} \tag{4.65}$$

$$[\frac{1}{w^4}\bar{v}_\alpha(\vec{p}_2,q)\gamma_{\nu_{\alpha\beta}}v_\beta(\vec{k}_1,r)\bar{v}_\gamma(\vec{p}_1,t)\gamma^\nu_{\gamma\delta}v_\delta(\vec{k}_2,s)\bar{v}_\epsilon(\vec{k}_1,r)\gamma_{\mu_{\epsilon\zeta}}v_\zeta(\vec{p}_2,q)\bar{v}_\eta(\vec{k}_2,s)\gamma^\mu_{\eta\theta}v_\theta(\vec{p}_1,t) \tag{4.66}$$

$$-\frac{1}{w^2h^2}\bar{v}_\alpha(\vec{p}_2,q)\gamma_{\nu_{\alpha\beta}}v_\beta(\vec{k}_1,r)\bar{v}_\gamma(\vec{p}_1,t)\gamma^\nu_{\gamma\delta}v_\delta(\vec{k}_2,s)\bar{v}_\epsilon(\vec{k}_1,r)\gamma_{\mu_{\epsilon\zeta}}v_\zeta(\vec{p}_1,t)\bar{v}_\eta(\vec{k}_2,s)\gamma^\mu_{\eta\theta}v_\theta(\vec{p}_2,q) \tag{4.67}$$

$$-\frac{1}{h^2w^2}\bar{v}_\alpha(\vec{p}_1,t)\gamma_{\nu_{\alpha\beta}}v_\beta(\vec{k}_1,r)\bar{v}_\gamma(\vec{p}_2,q)\gamma^\nu_{\gamma\delta}v_\delta(\vec{k}_2,s)\bar{v}_\epsilon(\vec{k}_1,r)\gamma_{\mu_{\epsilon\zeta}}v_\zeta(\vec{p}_2,q)\bar{v}_\eta(\vec{k}_2,s)\gamma^\mu_{\eta\theta}v_\theta(\vec{p}_1,t) \tag{4.68}$$

$$+\frac{1}{h^4}\bar{v}_\alpha(\vec{p}_1,t)\gamma_{\nu_{\alpha\beta}}v_\beta(\vec{k}_1,r)\bar{v}_\gamma(\vec{p}_2,q)\gamma^\nu_{\gamma\delta}v_\delta(\vec{k}_2,s)\bar{v}_\epsilon(\vec{k}_1,r)\gamma_{\mu_{\epsilon\zeta}}v_\zeta(\vec{p}_1,t)\bar{v}_\eta(\vec{k}_2,s)\gamma^\mu_{\eta\theta}v_\theta(\vec{p}_2,q)] \tag{4.69}$$

$$\tag{4.70}$$

$$|\mathcal{M}_{\text{unpol}}|^2 = \frac{e^4}{4}\sum_{t,q}\sum_{r,s} \tag{4.71}$$

$$[\frac{1}{w^4}v_\zeta(\vec{p}_2,q)\bar{v}_\alpha(\vec{p}_2,q)\gamma_{\nu_{\alpha\beta}}v_\beta(\vec{k}_1,r)\bar{v}_\epsilon(\vec{k}_1,r)v_\theta(\vec{p}_1,t)\bar{v}_\gamma(\vec{p}_1,t)\gamma^\nu_{\gamma\delta}v_\delta(\vec{k}_2,s)\bar{v}_\eta(\vec{k}_2,s)\gamma_{\mu_{\epsilon\zeta}}\gamma^\mu_{\eta\theta} \tag{4.72}$$

$$-\frac{1}{w^2h^2}v_\theta(\vec{p}_2,q)\bar{v}_\alpha(\vec{p}_2,q)\gamma_{\nu_{\alpha\beta}}v_\zeta(\vec{p}_1,t)\bar{v}_\gamma(\vec{p}_1,t)\gamma^\nu_{\gamma\delta}v_\beta(\vec{k}_1,r)\bar{v}_\epsilon(\vec{k}_1,r)\gamma_{\mu_{\epsilon\zeta}}v_\delta(\vec{k}_2,s)\bar{v}_\eta(\vec{k}_2,s)\gamma^\mu_{\eta\theta} \tag{4.73}$$

$$-\frac{1}{h^2w^2}\gamma_{\nu_{\alpha\beta}}v_\beta(\vec{k}_1,r)\bar{v}_\epsilon(\vec{k}_1,r)\gamma^\nu_{\gamma\delta}v_\delta(\vec{k}_2,s)\bar{v}_\eta(\vec{k}_2,s)\gamma_{\mu_{\epsilon\zeta}}v_\zeta(\vec{p}_2,q)\bar{v}_\gamma(\vec{p}_2,q)\gamma^\mu_{\eta\theta}v_\theta(\vec{p}_1,t)\bar{v}_\alpha(\vec{p}_1,t) \tag{4.74}$$

$$+\frac{1}{h^4}\gamma_{\nu_{\alpha\beta}}\gamma^\nu_{\gamma\delta}v_\delta(\vec{k}_2,s)\bar{v}_\eta(\vec{k}_2,s)v_\beta(\vec{k}_1,r)\bar{v}_\epsilon(\vec{k}_1,r)\gamma_{\mu_{\epsilon\zeta}}v_\zeta(\vec{p}_1,t)\bar{v}_\alpha(\vec{p}_1,t)\gamma^\mu_{\eta\theta}v_\theta(\vec{p}_2,q)\bar{v}_\gamma(\vec{p}_2,q)] \tag{4.75}$$

$$=\frac{e^4}{4}[\frac{1}{w^4}(p_2^\beta\gamma_\beta-m)_{\zeta\alpha}\gamma_{\nu_{\alpha\beta}}(k_1^\beta\gamma_\beta-m)_{\beta\epsilon}(p_1^\beta\gamma_\beta-m)_{\theta\gamma}\gamma^\nu_{\gamma\delta}(k_2^\beta\gamma_\beta-m)_{\delta\eta}\gamma_{\mu_{\epsilon\zeta}}\gamma^\mu_{\eta\theta} \tag{4.76}$$

$$-\frac{1}{w^2h^2}(p_2^\beta\gamma_\beta-m)_{\theta\alpha}\gamma_{\nu_{\alpha\beta}}(p_1^\beta\gamma_\beta-m)_{\zeta\gamma}\gamma^\nu_{\gamma\delta}(k_1^\beta\gamma_\beta-m)_{\beta\epsilon}\gamma_{\mu_{\epsilon\zeta}}(k_2^\beta\gamma_\beta-m)_{\delta\eta}\gamma^\mu_{\eta\theta} \tag{4.77}$$

$$-\frac{1}{h^2w^2}\gamma_{\nu_{\alpha\beta}}(k_1^\beta\gamma_\beta-m)_{\beta\epsilon}\gamma^\nu_{\gamma\delta}(k_2^\beta\gamma_\beta-m)_{\delta\eta}\gamma_{\mu_{\epsilon\zeta}}(p_2^\beta\gamma_\beta-m)_{\zeta\gamma}\gamma^\mu_{\eta\theta}(p_1^\beta\gamma_\beta-m)_{\theta\alpha} \tag{4.78}$$

$$+\frac{1}{h^4}\gamma_{\nu_{\alpha\beta}}\gamma^\nu_{\gamma\delta}(k_2^\beta\gamma_\beta-m)_{\delta\eta}(k_1^\beta\gamma_\beta-m)_{\beta\epsilon}\gamma_{\mu_{\epsilon\zeta}}(p_1^\beta\gamma_\beta-m)_{\zeta\alpha}\gamma^\mu_{\eta\theta}(p_2^\beta\gamma_\beta-m)_{\theta\gamma}]= \tag{4.79}$$

$$=\frac{e^4}{4}[\frac{1}{w^4}(p_2^\beta\gamma_\beta-m)_{\zeta\alpha}\gamma_{\nu_{\alpha\beta}}(k_1^\beta\gamma_\beta-m)_{\beta\epsilon}\gamma_{\mu_{\epsilon\zeta}}(p_1^\beta\gamma_\beta-m)_{\theta\gamma}\gamma^\nu_{\gamma\delta}(k_2^\beta\gamma_\beta-m)_{\delta\eta}\gamma^\mu_{\eta\theta} \tag{4.80}$$

$$-\frac{1}{w^2h^2}(p_2^\beta\gamma_\beta-m)_{\theta\alpha}\gamma_{\nu_{\alpha\beta}}(k_1^\beta\gamma_\beta-m)_{\beta\epsilon}\gamma_{\mu_{\epsilon\zeta}}(p_1^\beta\gamma_\beta-m)_{\zeta\gamma}\gamma^\nu_{\gamma\delta}(k_2^\beta\gamma_\beta-m)_{\delta\eta}\gamma^\mu_{\eta\theta} \tag{4.81}$$

$$-\frac{1}{h^2w^2}(k_1^\beta\gamma_\beta-m)_{\beta\epsilon}\gamma_{\mu_{\epsilon\zeta}}(p_2^\beta\gamma_\beta-m)_{\zeta\gamma}\gamma^\nu_{\gamma\delta}(k_2^\beta\gamma_\beta-m)_{\delta\eta}\gamma^\mu_{\eta\theta}(p_1^\beta\gamma_\beta-m)_{\theta\alpha}\gamma_{\nu_{\alpha\beta}} \tag{4.82}$$

$$+\frac{1}{h^4}(p_2^\beta\gamma_\beta-m)_{\theta\gamma}\gamma^\nu_{\gamma\delta}(k_2^\beta\gamma_\beta-m)_{\delta\eta}\gamma^\mu_{\eta\theta}(k_1^\beta\gamma_\beta-m)_{\beta\epsilon}\gamma_{\mu_{\epsilon\zeta}}(p_1^\beta\gamma_\beta-m)_{\zeta\alpha}\gamma_{\nu_{\alpha\beta}}] \tag{4.83}$$

We see by the indexes that we can form traces:

$$|\mathcal{M}_{\text{unpol}}|^2 = \frac{e^4}{4}[\frac{1}{w^4}\text{Tr}\left((p_2^\beta\gamma_\beta - m)\gamma_\nu(k_1^\beta\gamma_\beta - m)\gamma_\mu\right)\text{Tr}\left((p_1^\beta\gamma_\beta - m)\gamma^\nu(k_2^\beta\gamma_\beta - m)\gamma^\mu\right) \tag{4.84}$$

$$-\frac{1}{w^2h^2}\text{Tr}\left((p_2^\beta\gamma_\beta - m)\gamma_\nu(k_1^\beta\gamma_\beta - m)\gamma_\mu(p_1^\beta\gamma_\beta - m)\gamma^\nu(k_2^\beta\gamma_\beta - m)\gamma^\mu\right) \tag{4.85}$$

$$-\frac{1}{h^2w^2}\text{Tr}\left((k_1^\beta\gamma_\beta - m)\gamma_\mu(p_2^\beta\gamma_\beta - m)\gamma^\nu(k_2^\beta\gamma_\beta - m)\gamma^\mu(p_1^\beta\gamma_\beta - m)\gamma_\nu\right) \tag{4.86}$$

$$+\frac{1}{h^4}\text{Tr}\left((p_2^\beta\gamma_\beta - m)\gamma^\nu(k_2^\beta\gamma_\beta - m)\gamma^\mu\right)\text{Tr}\left((k_1^\beta\gamma_\beta - m)\gamma_\mu(p_1^\beta\gamma_\beta - m)\gamma_\nu\right)] \tag{4.87}$$

We can analize each of the previous 4 terms.
Let's start with the first:

$$\text{Tr}\left((p_2^\alpha\gamma_\alpha - m)\gamma_\nu(k_1^\beta\gamma_\beta - m)\gamma_\mu\right)\text{Tr}\left((p_1^\delta\gamma_\delta - m)\gamma^\nu(k_2^\eta\gamma_\eta - m)\gamma^\mu\right) = \tag{4.88}$$

$$\text{Tr}\left((p_{2_\alpha}\gamma^\alpha - m)\gamma_\nu(k_{1_\beta}\gamma^\beta - m)\gamma_\mu\right)\text{Tr}\left((p_{1_\delta}\gamma^\delta - m)\gamma^\nu(k_{2_\eta}\gamma^\eta - m)\gamma^\mu\right) = \tag{4.89}$$

$$\eta_{\nu\zeta}\eta_{\mu\theta}\text{Tr}\left((p_{2_\alpha}\gamma^\alpha - m)\gamma^\zeta(k_{1_\beta}\gamma^\beta - m)\gamma^\theta\right)\text{Tr}\left((p_{1_\delta}\gamma^\delta - m)\gamma^\nu(k_{2_\eta}\gamma^\eta - m)\gamma^\mu\right) \tag{4.90}$$

We need to compute only the first of these traces. After, we can obtain the second trace from the first by changing variable names.

We are going to use the following properties:

- Trace of any product of an odd number of γ^μ is zero.
- $\text{tr}\,(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu}$
- $\text{tr}\,(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4\,(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho})$

$$\text{Tr}\left((p_{2_\alpha}\gamma^\alpha - m)\gamma^\zeta(k_{1_\beta}\gamma^\beta - m)\gamma^\theta\right) = p_{2_\alpha}k_{1_\beta}\text{Tr}\left(\gamma^\alpha\gamma^\zeta\gamma^\beta\gamma^\theta\right) + m^2\text{Tr}\left(\gamma^\zeta\gamma^\theta\right) = \tag{4.91}$$

$$= p_{2_\alpha}k_{1_\beta}4\left(\eta^{\alpha\zeta}\eta^{\beta\theta} - \eta^{\alpha\beta}\eta^{\zeta\theta} + \eta^{\alpha\theta}\eta^{\zeta\beta}\right) + m^2 4\eta^{\zeta\theta} = \tag{4.92}$$

$$= 4\left(p_2^\zeta k_1^\theta - p_{2_\alpha}k_1^\alpha\eta^{\zeta\theta} + p_2^\theta k_1^\zeta\right) + 4m^2\eta^{\zeta\theta} \tag{4.93}$$

The second trace of 4.90 is of the same form of the first therefore we need just to change the variable names:

$$\zeta \to \nu \qquad \theta \to \mu \qquad p_2 \to p_1 \qquad k_1 \to k_2 \tag{4.94}$$

$$\text{Tr}\left((p_{1_\delta}\gamma^\delta - m)\gamma^\nu(k_{2_\eta}\gamma^\eta - m)\gamma^\mu\right) = 4\,(p_1^\nu k_2^\mu - p_{1_\alpha}k_2^\alpha\eta^{\nu\mu} + p_1^\mu k_2^\nu) + 4m^2\eta^{\nu\mu} \tag{4.95}$$

Putting these results into 4.90:

$$\mathrm{Tr}\left((p_2^\alpha\gamma_\alpha - m)\gamma_\nu(k_1^\beta\gamma_\beta - m)\gamma_\mu\right)\mathrm{Tr}\left((p_1^\delta\gamma_\delta - m)\gamma^\nu(k_2^\eta\gamma_\eta - m)\gamma^\mu\right) = \tag{4.96}$$

$$= \eta_{\nu\zeta}\eta_{\mu\theta}\left[4\left(p_2^\zeta k_1^\theta - p_{2_\alpha}k_1^\alpha\eta^{\zeta\theta} + p_2^\theta k_1^\zeta\right) + 4m^2\eta^{\zeta\theta}\right]\left[4\left(p_1^\nu k_2^\mu - p_{1_\alpha}k_2^\alpha\eta^{\nu\mu} + p_1^\mu k_2^\nu\right) + 4m^2\eta^{\nu\mu}\right] = \tag{4.97}$$

$$= 16\left[\left(p_{2_\nu}k_{1_\mu} - p_{2_\alpha}k_1^\alpha\eta_{\nu\mu} + p_{2_\mu}k_{1_\nu}\right) + m^2\eta_{\nu\mu}\right]\left[\left(p_1^\nu k_2^\mu - p_{1_\alpha}k_2^\alpha\eta^{\nu\mu} + p_1^\mu k_2^\nu\right) + m^2\eta^{\nu\mu}\right] = \tag{4.98}$$

$$= 16[(p_2p_1)(k_1k_2) - (p_2k_1)(p_1k_2) + (p_2k_2)(p_1k_1) + m^2(p_2k_1) \tag{4.99}$$

$$- (p_2k_1)(p_1k_2) + 4(p_2k_1)(p_1k_2) - (p_2k_1)(p_1k_2) - 4m^2(p_2k_1) \tag{4.100}$$

$$+ (p_2k_2)(k_1p_1) - (p_2k_1)(p_1k_2) + (p_2p_1)(k_1k_2) + m^2(p_2k_1) \tag{4.101}$$

$$+ m^2(p_1k_2) - 4m^2(p_1k_2) + m^2(p_1k_2) + 4m^2] = \tag{4.102}$$

$$= 32[(p_2p_1)(k_1k_2) + (p_2k_2)(k_1p_1) - m^2(p_2k_1) - m^2(p_1k_2) + 2m^4] \tag{4.103}$$

Let's now analize the fourth term which appear in the equation of $|\mathcal{M}_{unpol}|^2$ 4.87.
If we put it in the following form :

$$\mathrm{Tr}\left((p_2^\beta\gamma_\beta - m)\gamma^\nu(k_2^\beta\gamma_\beta - m)\gamma^\mu\right)\mathrm{Tr}\left((k_1^\beta\gamma_\beta - m)\gamma_\mu(p_1^\beta\gamma_\beta - m)\gamma_\nu\right)] = \tag{4.104}$$

$$= \mathrm{Tr}\left((p_1^\beta\gamma_\beta - m)\gamma_\nu(k_1^\beta\gamma_\beta - m)\gamma_\mu\right)\mathrm{Tr}\left((p_2^\beta\gamma_\beta - m)\gamma^\nu(k_2^\beta\gamma_\beta - m)\gamma^\mu\right)] \tag{4.105}$$

And we do the change of variables:

$$p_1 \leftrightarrow p_2 \tag{4.106}$$

We obtain the same form of the first term. So we can write:

$$\mathrm{Tr}\left((p_2^\beta\gamma_\beta - m)\gamma^\nu(k_2^\beta\gamma_\beta - m)\gamma^\mu\right)\mathrm{Tr}\left((k_1^\beta\gamma_\beta - m)\gamma_\mu(p_1^\beta\gamma_\beta - m)\gamma_\nu\right) = \tag{4.107}$$

$$= 32[(p_1p_2)(k_1k_2) + (p_1k_2)(k_1p_2) - m^2(p_1k_1) - m^2(p_2k_2) + 2m^4] \tag{4.108}$$

Let's compute the second term that appears in the the equation of $|\mathcal{M}_{unpol}|^2$.
In the following we are going to use the fact that trace of any product of an odd number of γ^μ is zero.

$$\mathrm{Tr}\left((p_2^\beta\gamma_\beta - m)\gamma_\nu(k_1^\beta\gamma_\beta - m)\gamma_\mu(p_1^\beta\gamma_\beta - m)\gamma^\nu(k_2^\beta\gamma_\beta - m)\gamma^\mu\right) = \tag{4.109}$$

$$= \mathrm{Tr}\left((p_{2_\alpha}\gamma^\alpha - m)\gamma_\nu(k_{1_\beta}\gamma^\beta - m)\gamma_\mu(p_{1_\delta}\gamma^\delta - m)\gamma^\nu(k_{2_\zeta}\gamma^\zeta - m)\gamma^\mu\right) = \tag{4.110}$$

$$= p_{2_\alpha}k_{1_\beta}p_{1_\delta}k_{2_\zeta}\mathrm{Tr}\left(\gamma^\alpha\gamma_\nu\gamma^\beta\gamma_\mu\gamma^\delta\gamma^\nu\gamma^\zeta\gamma^\mu\right) + m^2p_{2_\alpha}k_{1_\beta}\mathrm{Tr}\left(\gamma^\alpha\gamma_\nu\gamma^\beta\gamma_\mu\gamma^\nu\gamma^\mu\right) \tag{4.111}$$

$$+ m^2p_{2_\alpha}k_{2_\zeta}\mathrm{Tr}\left(\gamma^\alpha\gamma_\nu\gamma_\mu\gamma^\nu\gamma^\zeta\gamma^\mu\right) + m^2p_{2_\alpha}p_{1_\delta}\mathrm{Tr}\left(\gamma^\alpha\gamma_\nu\gamma_\mu\gamma^\delta\gamma^\nu\gamma^\mu\right) + \tag{4.112}$$

$$+ m^2k_{1_\beta}p_{1_\delta}\mathrm{Tr}\left(\gamma_\nu\gamma^\beta\gamma_\mu\gamma^\delta\gamma^\nu\gamma^\mu\right) + m^2k_{1_\beta}k_{2_\zeta}\mathrm{Tr}\left(\gamma_\nu\gamma^\beta\gamma_\mu\gamma^\nu\gamma^\zeta\gamma^\mu\right) + \tag{4.113}$$

$$+ m^2p_{1_\delta}k_{2_\zeta}\mathrm{Tr}\left(\gamma_\nu\gamma_\mu\gamma^\delta\gamma^\nu\gamma^\zeta\gamma^\mu\right) + m^4\mathrm{Tr}\left(\gamma_\nu\gamma_\mu\gamma^\nu\gamma^\mu\right) \tag{4.114}$$

In order to compute these traces, we use:

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu$$
$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4\eta^{\nu\rho}I_4$$
$$\mathrm{tr}\left(\gamma^\mu\gamma^\nu\right) = 4\eta^{\mu\nu}$$
$$\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu$$
$$\gamma^\mu\gamma_\mu = 4I_4$$

Hence:

- $\mathrm{Tr}\left(\gamma^\alpha\gamma_\nu\gamma^\beta\gamma_\mu\gamma^\delta\gamma^\nu\gamma^\zeta\gamma^\mu\right) = -32\eta^{\beta\zeta}\eta^{\alpha\delta}$

  Proof.

  $$\mathrm{Tr}\left(\gamma^\alpha\gamma_\nu\gamma^\beta\gamma^\mu\gamma^\delta\gamma^\nu\gamma^\zeta\gamma_\mu\right) = -2\mathrm{Tr}\left(\gamma^\alpha\gamma_\nu\gamma^\beta\gamma^\zeta\gamma^\nu\gamma^\delta\right) = -2\mathrm{Tr}\left(\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^\zeta\gamma_\nu\gamma^\delta\right) =$$
  $$= -8\mathrm{Tr}\left(\gamma^\alpha\eta^{\beta\zeta}\gamma^\delta\right) = -32\eta^{\beta\zeta}\eta^{\alpha\delta}$$

  ■

- $\mathrm{Tr}\left(\gamma^\alpha\gamma_\nu\gamma^\beta\gamma_\mu\gamma^\nu\gamma^\mu\right) = 16\eta^{\alpha\beta}$

  Proof.

  $$\mathrm{Tr}\left(\gamma^\alpha\gamma_\nu\gamma^\beta\gamma_\mu\gamma^\nu\gamma^\mu\right) = \mathrm{Tr}\left(\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^\mu\gamma_\nu\gamma_\mu\right) = 4\mathrm{Tr}\left(\gamma^\alpha\eta^{\beta\mu}\gamma_\mu\right) = 16\eta^{\alpha\beta}$$

  ■

- $\mathrm{Tr}\left(\gamma^\alpha\gamma_\nu\gamma_\mu\gamma^\nu\gamma^\zeta\gamma^\mu\right) = 16\eta^{\alpha\zeta}$ and similar could be shown in the same way of previous one

- $\mathrm{Tr}\left(\gamma_\nu\gamma_\mu\gamma^\nu\gamma^\mu\right) = -32$

  Proof.

  $$\mathrm{Tr}\left(\gamma_\nu\gamma_\mu\gamma^\nu\gamma^\mu\right) = \mathrm{Tr}\left(\gamma^\nu\gamma^\mu\gamma_\nu\gamma_\mu\right) = -2\mathrm{Tr}\left(\gamma^\mu\gamma_\mu\right) = -2\mathrm{Tr}\left(\gamma^\mu\gamma_\mu\right) = -8\mathrm{Tr}\left(I_4\right) = -32$$

  ■

Therefore:

$$\mathrm{Tr}\left((p_2^\beta\gamma_\beta - m)\gamma_\nu(k_1^\beta\gamma_\beta - m)\gamma_\mu(p_1^\beta\gamma_\beta - m)\gamma^\nu(k_2^\beta\gamma_\beta - m)\gamma^\mu\right) = \tag{4.115}$$
$$= -32p_{2_\alpha}k_{1_\beta}p_{1_\delta}k_{2_\zeta}\eta^{\beta\zeta}\eta^{\alpha\delta} + 16m^2p_{2_\alpha}k_{1_\beta}\eta^{\alpha\beta} + 16m^2p_{2_\alpha}k_{2_\zeta}\eta^{\alpha\zeta} + 16m^2p_{2_\alpha}p_{1_\delta}\eta^{\alpha\delta} \tag{4.116}$$
$$+ 16m^2k_{1_\beta}p_{1_\delta}\eta^{\beta\delta} + 16m^2k_{1_\beta}k_{2_\zeta}\eta^{\beta\zeta} + 16m^2p_{1_\delta}k_{2_\zeta}\eta^{\delta\zeta} - 32m^4 = \tag{4.117}$$
$$= -32(p_2p_1)(k_1k_2) + 16m^2p_2k_1 + 16m^2p_2k_2 + 16m^2p_2p_1 + 16m^2k_1p_1 \tag{4.118}$$
$$+ 16m^2k_1k_2 + 16m^2p_1k_2 - 32m^4 \tag{4.119}$$

Let's now analize the third term which appear in the equation of $\|\mathcal{M}_{unpol}|^2$ 4.87. If we put in the following form :

$$\mathrm{Tr}\left((k_1^\beta\gamma_\beta - m)\gamma_\mu(p_2^\beta\gamma_\beta - m)\gamma^\nu(k_2^\beta\gamma_\beta - m)\gamma^\mu(p_1^\beta\gamma_\beta - m)\gamma_\nu\right) = \tag{4.120}$$
$$\mathrm{Tr}\left((p_1^\beta\gamma_\beta - m)\gamma_\nu(k_1^\beta\gamma_\beta - m)\gamma_\mu(p_2^\beta\gamma_\beta - m)\gamma^\nu(k_2^\beta\gamma_\beta - m)\gamma^\mu\right) = \tag{4.121}$$
$$\tag{4.122}$$

And we do the change of variables:

$$p_1 \leftrightarrow p_2 \tag{4.123}$$

We obtain the same form of the second term. So we can write:

$$\mathrm{Tr}\left((k_1^\beta\gamma_\beta - m)\gamma_\mu(p_2^\beta\gamma_\beta - m)\gamma^\nu(k_2^\beta\gamma_\beta - m)\gamma^\mu(p_1^\beta\gamma_\beta - m)\gamma_\nu\right) = \tag{4.124}$$

$$= -32(p_1p_2)(k_1k_2) + 16m^2p_1k_1 + 16m^2p_1k_2 + 16m^2p_1p_2 + 16m^2k_1p_2 \tag{4.125}$$

$$+ 16m^2k_1k_2 + 16m^2p_2k_2 - 32m^4 \tag{4.126}$$

Finally, we can write:

$$|\mathcal{M}_{\text{unpol}}|^2 = \frac{e^4}{4}[\frac{1}{w^4}\mathrm{Tr}\left((p_2^\beta\gamma_\beta - m)\gamma_\nu(k_1^\beta\gamma_\beta - m)\gamma_\mu\right)\mathrm{Tr}\left((p_1^\beta\gamma_\beta - m)\gamma^\nu(k_2^\beta\gamma_\beta - m)\gamma^\mu\right) \tag{4.127}$$

$$-\frac{1}{w^2h^2}\mathrm{Tr}\left((p_2^\beta\gamma_\beta - m)\gamma_\nu(k_1^\beta\gamma_\beta - m)\gamma_\mu(p_1^\beta\gamma_\beta - m)\gamma^\nu(k_2^\beta\gamma_\beta - m)\gamma^\mu\right) \tag{4.128}$$

$$-\frac{1}{h^2w^2}\mathrm{Tr}\left((k_1^\beta\gamma_\beta - m)\gamma_\mu(p_2^\beta\gamma_\beta - m)\gamma^\nu(k_2^\beta\gamma_\beta - m)\gamma^\mu(p_1^\beta\gamma_\beta - m)\gamma_\nu\right) \tag{4.129}$$

$$+\frac{1}{h^4}\mathrm{Tr}\left((p_2^\beta\gamma_\beta - m)\gamma^\nu(k_2^\beta\gamma_\beta - m)\gamma^\mu\right)\mathrm{Tr}\left((k_1^\beta\gamma_\beta - m)\gamma_\mu(p_1^\beta\gamma_\beta - m)\gamma_\nu\right)] = \tag{4.130}$$

$$= \frac{32e^4}{4}\{\frac{1}{w^4}[(p_2p_1)(k_1k_2) + (p_2k_2)(k_1p_1) - m^2(p_2k_1) - m^2(p_1k_2) + 2m^4] \tag{4.131}$$

$$-\frac{1}{w^2h^2}[-2(p_2p_1)(k_1k_2) + m^2p_2k_1 + m^2p_2k_2 + m^2p_2p_1 + m^2k_1p_1 + m^2k_1k_2 + m^2p_1k_2 - 2m^4] \tag{4.132}$$

$$+\frac{1}{h^4}[(p_1p_2)(k_1k_2) + (p_1k_2)(k_1p_2) - m^2(p_1k_1) - m^2(p_2k_2) + 2m^4]\} \tag{4.133}$$

$$\tag{4.134}$$

We can introduce Mandelstam variables to rewrite the previous result:

$$s = (k_1 + k_2)^2 = (p_1 + p_2)^2 \tag{4.135}$$

$$t = (p_1 - k_1)^2 = (p_2 - k_2)^2 \tag{4.136}$$

$$u = (k_2 - p_1)^2 = (k_1 - p_2)^2 \tag{4.137}$$

Hence, we have:

$$h^2 = (k_1 - p_1)^2 = t \quad , \quad w^2 = (k_1 - p_2)^2 = u \tag{4.138}$$

$$k_1k_2 = \frac{s - 2m^2}{2} = p_1p_2 \tag{4.139}$$

Proof.

$$s = (k_1 + k_2)^2 = k_1^2 + k_2^2 + 2k_1k_2 = 2m^2 + 2k_1k_2 \tag{4.140}$$

■

$$k_1p_1 = \frac{2m^2 - t}{2} = k_2p_2 \tag{4.141}$$

Proof.

$$t = (k_1 - p_1)^2 = k_1^2 + p_1^2 - 2k_1p_1 = 2m^2 - 2k_1p_1 \tag{4.142}$$

■

$$k_2 p_1 = \frac{2m^2 - u}{2} = k_1 p_2 \tag{4.143}$$

Proof.

$$t = (k_2 - p_1)^2 = k_2^2 + p_1^2 - 2k_2 p_1 = 2m^2 - 2k_1 p_1 \tag{4.144}$$

■

Hence:

$$|\mathcal{M}_{\text{unpol}}|^2 = 8e^4\{\frac{1}{u^2}[(\frac{s-2m^2}{2})^2 + (\frac{2m^2-t}{2})^2 - 2m^2(\frac{2m^2-u}{2}) + 2m^4]$$
$$-\frac{1}{ut}[-2(\frac{s-2m^2}{2})^2 + 2m^2(\frac{2m^2-u}{2}) + 2m^2(\frac{2m^2-t}{2}) + 2m^2(\frac{s-2m^2}{2}) - 2m^4]$$
$$+\frac{1}{t^2}[(\frac{s-2m^2}{2})^2 + (\frac{2m^2-u}{2})^2 - 2m^2(\frac{2m^2-t}{2}) + 2m^4]\}$$

Using the relation:

$$s + t + u = 4m^2 \tag{4.145}$$

we can put the previous equation in the form :

$$|\mathcal{M}_{\text{unpol}}|^2 = 2e^4 \left\{ \frac{1}{t^2}\left(s^2 + u^2 - 8m^2(s+u) + 24m^4\right) \right. \tag{4.146}$$

$$\left. + \frac{1}{u^2}\left(s^2 + t^2 - 8m^2(s+t) + 24m^4\right) + \frac{2}{tu}\left(s^2 - 8m^2 s + 12m^4\right) \right\} \tag{4.147}$$

In the center of mass frame we have:

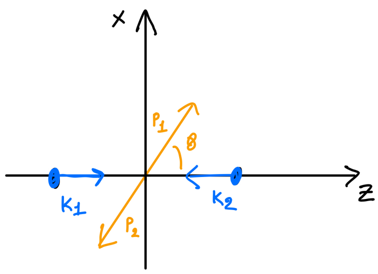

Figure 3: particles scattering in the CM frame

$\vec{P}_{tot} = \vec{p}_1 + \vec{p}_2 = \vec{k}_1 + \vec{k}_2 = 0 \longrightarrow \vec{p} \equiv \vec{p}_1 = -\vec{p}_2 \ , \ \vec{k} \equiv \vec{k}_1 = -\vec{k}_2$

Since the particles have the same mass and momentum they have also the same energy for on shell relation.

$$|\vec{k}| = \sqrt{E^2 - m^2} \tag{4.148}$$

$$k_1 = (E, 0, 0, \sqrt{E^2 - m^2}) \tag{4.149}$$

$$k_2 = (E, 0, 0, -\sqrt{E^2 - m^2}) \tag{4.150}$$

$$p_1 = (E', \sqrt{E'^2 - m^2}\sin(\theta), 0, \sqrt{E'^2 - m^2}\cos(\theta)) \tag{4.151}$$

$$p_2 = (E', -\sqrt{E'^2 - m^2}\sin(\theta), 0, -\sqrt{E'^2 - m^2}\cos(\theta)) \tag{4.152}$$

Tetramomentum conservation imposes $E' = E$.
Hence we have all variables determined by (E, θ):

$$k_1 = (E, 0, 0, \sqrt{E^2 - m^2}) \tag{4.153}$$

$$k_2 = (E, 0, 0, -\sqrt{E^2 - m^2}) \tag{4.154}$$

$$p_1 = (E, \sqrt{E^2 - m^2}\sin(\theta), 0, \sqrt{E^2 - m^2}\cos(\theta)) \tag{4.155}$$

$$p_2 = (E, -\sqrt{E^2 - m^2}\sin(\theta), 0, -\sqrt{E^2 - m^2}\cos(\theta)) \tag{4.156}$$

Let's rewrite the Mandelstam variables:

$$s = 4E^2 = E_{CM}^2 \tag{4.157}$$

Proof. $s = (k_1 + k_2)^2 = 4E^2 = E_{CM}^2$ ■

$$t = -2|\vec{p}|^2(1 - \cos(\theta)) \tag{4.158}$$

Proof.

$$t = (p_1 - k_1)^2 = 2m^2 - 2\left(E, \sqrt{E^2 - m^2}\sin(\theta), 0, \sqrt{E^2 - m^2}\cos(\theta)\right)\left(E, 0, 0, \sqrt{E^2 - m^2}\right) \tag{4.159}$$

$$= 2m^2 - 2(E^2 - (E^2 - m^2)\cos(\theta)) = 2(m^2 - E^2) + 2(E^2 - m^2)\cos(\theta) = \tag{4.160}$$

$$= 2(m^2 - E^2)(1 - \cos(\theta)) = -2|\vec{p}|^2(1 - \cos(\theta)) \tag{4.161}$$

$$\tag{4.162}$$

■

$$u = -2|\vec{p}|^2(1 + \cos(\theta)) \tag{4.163}$$

Proof.

$$u = (k_2 - p_1)^2 = 2m^2 - 2k_2p_1 = 2m^2 - 2(E^2 + (E^2 - m^2)\cos(\theta)) = \tag{4.164}$$

$$= 2(m^2 - E^2) + 2(m^2 - E^2)\cos(\theta) = 2(m^2 - E^2)(1 + \cos(\theta)) = -2|\vec{p}|^2(1 + \cos(\theta)) \tag{4.165}$$

$$\tag{4.166}$$

■

Substituting in eq(4.147):

$$|\mathcal{M}_{\text{unpol}}|^2 = 2e^4 \left\{ \frac{1}{t^2}\left(s^2+u^2-8m^2(s+u)+24m^4\right) + \frac{1}{u^2}\left(s^2+t^2-8m^2(s+t)+24m^4\right) \right.$$
$$\left. + \frac{2}{tu}\left(s^2-8m^2s+12m^4\right) \right\} =$$
$$2e^4 \left\{ \frac{1}{4|\vec{p}|^4(1-\cos\theta)^2}\left(16E^4+4|\vec{p}|^4(1+\cos\theta)^2-8m^2(4E^2-2|\vec{p}|^2(1+\cos\theta))+24m^4\right) \right.$$
$$+ \frac{1}{4|\vec{p}|^4(1+\cos\theta)^2}\left(16E^4+4|\vec{p}|^4(1-\cos\theta)^2-8m^2\left(4E^2-2|\vec{p}|^2(1-\cos\theta)\right)+24m^4\right)$$
$$\left. + \frac{2}{4|\vec{p}|^4\sin^2\theta}\left(16E^4-32m^2E^2+12m^4\right) \right\}$$

The differential cross section in the center of mass frame could be computed using:

$$d\sigma = \frac{d^3p_1}{(2\pi)^3 2E_{p_1}} \frac{d^3p_2}{(2\pi)^3 2E_{p_2}} \frac{(2\pi)^4\delta^{(4)}(k_1+k_2-p_1-p_2)}{4\sqrt{(k_1k_2)^2-m^4}} |\mathcal{M}_{unpol}(E_{p_1},\theta_1)|^2 \tag{4.167}$$

We have:

$$\sqrt{(k_1k_2)^2-m^4} = 2E\sqrt{(E^2-m^2)} \tag{4.168}$$

Proof.

$$k_1k_2 = \frac{-2m^2+s}{2} = 2E^2-m^2 \tag{4.169}$$
$$(k_1k_2)^2-m^4 = 4E^4-4E^2m^2 \tag{4.170}$$

∎

We are in the C.M. frame, hence:

$$\vec{k} \equiv \vec{k}_1 = -\vec{k}_2 \quad , \quad E \equiv E_{k_1} = \sqrt{m^2+|\vec{k}|^2} = E_{k_2} \tag{4.171}$$

$$\delta^{(4)}(k_1+k_2-p_1-p_2) = \delta\left(E_{k_1}+E_{k_2}-E_{p_1}-E_{p_2}\right)\delta^{(3)}\left(\vec{k}_1+\vec{k}_2-\vec{p}_1-\vec{p}_2\right) \tag{4.172}$$
$$= \delta\left(E_{k_1}+E_{k_2}-E_{p_1}-E_{p_2}\right)\delta^{(3)}\left(-\vec{p}_1-\vec{p}_2\right) \tag{4.173}$$

Hence:

$$d\sigma = \frac{d^3p_1}{(2\pi)^3 2E_{p_1}} \int \frac{d^3p_2}{(2\pi)^3 2E_{p_2}} \frac{(2\pi)^4\delta\left(E_{k_1}+E_{k_2}-E_{p_1}-E_{p_2}\right)\delta^{(3)}\left(-\vec{p}_1-\vec{p}_2\right)}{8E\sqrt{(E^2-m^2)}} |\mathcal{M}_{unpol}|^2 =$$
$$= \frac{d^3p_1}{(2\pi)^3 2E_{p_1}} \frac{1}{(2\pi)^3 2E_{p_1}} \frac{(2\pi)^4\delta\left(2E-2\sqrt{m^2+|\vec{p}_1|^2}\right)}{8E\sqrt{(E^2-m^2)}} |\mathcal{M}_{unpol}|^2$$

We have:

$$d^3p_1 = |\vec{p}_1|^2 d|\vec{p}_1| d\Omega \tag{4.174}$$

$$\frac{d\sigma}{d\Omega} = \int d|\vec{p}_1| \frac{|\vec{p}_1|^2}{(2\pi)^6 4E_{p_1}^2} \frac{(2\pi)^4 \delta\left(2E - 2\sqrt{m^2 + |\vec{p}_1|^2}\right)}{8E\sqrt{(E^2 - m^2)}} |\mathcal{M}_{unpol}|^2$$

We use the delta property:

$$\delta(f(x)) = \sum_{i=1}^{N} \frac{\delta\left(x - x_{0,i}\right)}{|f'\left(x_{0,i}\right)|} \tag{4.175}$$

where $\{x_{0,i}\}_{i=1,N}$ are the zeros of $f(x)$.

In our case :

$$f(|\vec{p}_1|) = 2E - 2\sqrt{m^2 + |\vec{p}_1|^2} \tag{4.176}$$

with zero:

$$|\vec{p}_{1,0}| = \sqrt{E^2 - m^2} \tag{4.177}$$

Hence:

$$f'(|\vec{p}_{1,0}|) = -\frac{2\,|\vec{p}_{1,0}|}{\sqrt{m^2 + |\vec{p}_{1,0}|^2}} = -2\frac{|\vec{p}_{1,0}|}{E_{p_{1,0}}} \tag{4.178}$$

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \int d|\vec{p}_1| \frac{|\vec{p}_1|^2}{(2\pi)^6 4E_{p_1}^2} \frac{(2\pi)^4 \delta\left(|\vec{p}_1| - |\vec{p}_{1,0}|\right)}{8E\sqrt{(E^2 - m^2)}2\frac{|\vec{p}_{1,0}|}{E_{p_{1,0}}}} |\mathcal{M}_{unpol}|^2 \\ &= \frac{|\vec{p}_{1,0}|}{(2\pi)^6 4E_{p_{1,0}}} \frac{(2\pi)^4}{16E\sqrt{(E^2 - m^2)}} |\mathcal{M}_{unpol}|^2 \\ &= \frac{1}{256\pi^2 E^2} |\mathcal{M}_{unpol}|^2\end{aligned}$$

Using $\alpha = \frac{e^2}{4\pi}$, we have:

$$\frac{d\sigma}{d\Omega} = \frac{1}{256\pi^2 E^2} |\mathcal{M}_{unpol}|^2 = \frac{\alpha^2}{8E^2} \tag{4.179}$$

$$\left\{\frac{1}{4|\vec{p}|^4(1-\cos\theta)^2}\left(16E^4 + 4|\vec{p}|^4(1+\cos\theta)^2 - 8m^2(4E^2 - 2|\vec{p}|^2(1+\cos\theta)) + 24m^4\right)\right. \tag{4.180}$$

$$+\frac{1}{4|\vec{p}|^4(1+\cos\theta)^2}\left(16E^4 + 4|\vec{p}|^4(1-\cos\theta)^2 - 8m^2\left(4E^2 - 2|\vec{p}|^2(1-\cos\theta)\right) + 24m^4\right) \tag{4.181}$$

$$\left. + \frac{2}{4|\vec{p}|^4\sin^2\theta}\left(16E^4 - 32m^2E^2 + 12m^4\right)\right\} \tag{4.182}$$

$$\tag{4.183}$$

In the non-relativistic $E = \sqrt{m^2 + |\vec{p}|^2} \simeq m$ (we have $|\vec{p}| \ll m$):

$$\frac{d\sigma}{d\Omega} \simeq \frac{\alpha^2 m^2}{4|\vec{p}|^4}\left\{\frac{4}{\sin^4\theta} - \frac{3}{\sin^2\theta}\right\} \tag{4.184}$$

Proof.

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{8m^2}\left\{\frac{1}{4|\vec{p}|^4(1-\cos\theta)^2}\left(16m^4 - 32m^4 + 24m^4\right)\right. \tag{4.185}$$

$$+\frac{1}{4|\vec{p}|^4(1+\cos\theta)^2}\left(16m^4 - 32m^4 + 24m^4\right) \tag{4.186}$$

$$\left.+\frac{2}{4|\vec{p}|^4\sin^2\theta}\left(16m^4 - 32m^4 + 12m^4\right)\right\} = \tag{4.187}$$

$$= \frac{\alpha^2 m^2}{4|\vec{p}|^4}\left\{\frac{1}{(1-\cos\theta)^2} + \frac{1}{(1+\cos\theta)^2} - \frac{1}{\sin^2\theta}\right\} = \tag{4.188}$$

$$= \frac{\alpha^2 m^2}{4|\vec{p}|^4}\left\{\frac{4}{\sin^4\theta} - \frac{3}{\sin^2\theta}\right\} \tag{4.189}$$

$$\tag{4.190}$$

■

In the ultra-relativistic limits $E = \sqrt{m^2 + |\vec{p}|^2} \simeq |\vec{p}|$ (we have $|\vec{p}| \gg m$):

$$\frac{d\sigma}{d\Omega} \simeq= \frac{\alpha^2}{4E^2\sin^4\theta}\left(3+\cos^2\theta\right)^2 \tag{4.191}$$

Proof.

$$\frac{d\sigma}{d\Omega} \simeq \frac{\alpha^2}{32E^2}\left\{\frac{1}{(1-\cos\theta)^2}\left(16 + 4(1+\cos\theta)^2\right) + \frac{1}{(1+\cos\theta)^2}\left(16 + 4(1-\cos\theta)^2\right) + \frac{32}{\sin^2\theta}\right\} \tag{4.192}$$

$$= \frac{\alpha^2}{32E^2\sin^4\theta}\left\{8\left(3+\cos^2\theta\right)^2\right\} = \frac{\alpha^2}{4E^2\sin^4\theta}\left(3+\cos^2\theta\right)^2 \tag{4.193}$$

$$\tag{4.194}$$

■