Projection-Valued Measure ← PROJECTIVE MEASUREMENTS

A PVM measurement is obtained through a projector system [1], which is defined as a set of operators $\{P_i, i \in M\}$ of Hilbert space H, where M is an alphabet set of all possible outcomes of the measurement, if these operators have properties: 1) P_i is Hermitian: $P_i = {P_i}^\dagger$; 2) P_i is positive semi-definite: $P_i \geq 0$; 3) P_i is idempotent: ${P_i}^2 = P_i$; 4) P_i is pairwise orthogonal: $P_i P_j = \delta_{ij} = 0$, for $i \neq j$; 5) $\{P_i, i \in M\}$ forms a resolution of the identity on H: $\sum_{i \in M} P_i = I_H$.

The probability of obtaining outcome i for a given state $s = |\psi\rangle$ is specified by

$$p_m(\mathrm{i}|\psi) = \mathrm{P}(\mathrm{m} = \mathrm{i}|s = |\psi\rangle) = \langle\psi|P_i|\psi\rangle \tag{1}$$

And the post-measurement state is given by

$$|\psi_{post}^{(i)}\rangle = \frac{P_i|\psi\rangle}{\sqrt{\langle\psi|P_i|\psi\rangle}} \tag{2}$$

For mixed state, specified by the density matrix ρ, the probability of obtaining outcome i is given by

$$p_m(\mathrm{i}|\rho) = \mathrm{tr}(P_i\rho) \tag{3}$$

where tr(·) is the trace operation. And the post-measurement state is specified by the following density matrix:

$$\rho_{post}^{(i)} = \frac{P_i\rho P_i}{\mathrm{tr}(P_i\rho)} \tag{4}$$

← normalization.

- POVM : set $\{E_a\}_{a=1}^{N}$ such that $\begin{cases} \bullet\, E_a \geq 0 \;;\; a=1,\dots,N \\ \bullet\, \sum_{a=1}^{N} E_a = \mathbb{1} \end{cases}$ ($\Rightarrow E_A^\dagger = E_A$)

- AXIOMS: Given a state $\rho \in D(\mathcal{H})$, If I perform a POVM (generalized measurement) on ρ,
  - the $\boxed{\text{probability}}$ of getting the outcome "a" is;

$$\mathrm{Prob}(a) = \mathrm{tr}(E_a \rho)$$

- The state after getting the measurement outcome "a" is not uniquely determined if we only know the POVM.
  We need something more like "the physical implementation" of this POVM.

NAIMARK DILATATION THEOREM:

- POVMs, also called generalized measurements, can be understood as projective measurements on a larger system.

- $\boxed{\text{POVM}}$: • $\{E_a\}_{a=1}^{N}$ such that : $\begin{cases} \bullet\ E_a \geq 0 \\ \bullet\ \sum_{a=1}^{N} E_a = 1 \end{cases}$

  • $\text{Prob}(a) = tr(\rho E_a)$

- $\boxed{\text{PVM}}$: • $\{\Pi_a\}_{a=1}^{N'}$ such that : $\begin{cases} \bullet\ \Pi_a \geq 0 \\ \bullet\ \sum_{a=1}^{N} \Pi_a = 1 \\ \bullet\ \Pi_a \Pi_b = \delta_{ab} \Pi_a \end{cases}$

  • $\text{Prob}(a) = tr(\rho \Pi_a)$

  • $\rho_a \Big|_{\substack{\text{after} \\ \text{seen "a"}}} = \dfrac{\Pi_a \rho \Pi_a}{tr(\Pi_a \rho)}$

- We want to show that given a POVM $\{E_K\}_{K=1}^{N}$ acting on $\mathcal{H}_A$, $\exists$ PVM $\{\Pi_K\}_{K=1}^{N}$ and a unitary U_{AB} acting on $\mathcal{H}_A \otimes \mathcal{H}_B$ such that:

$$\text{Prob}(K) = tr(E_K \rho) = tr\left(\Pi_K \left(U_{AB}\, \rho \otimes |0\rangle_B \langle 0|_B\, U_{AB}^{\dagger}\right)\right)$$

which means that the probability of measuring "K" using $\{E_a\}_{a=1}^{N}$ POVM, can be understood as the probability of measuring "K" performing a PVM $\{\Pi_a\}_{a=1}^{N}$ after evolving the system $\rho \otimes |0\rangle_B \langle 0|_B$ using a unitary U_{AB}.

- In particular we can show that we can choose an $\mathcal{H}_B$ of dimension N = size of POVM set.

  • $\Pi_K = \mathbb{1} \otimes |K\rangle_B \langle K|_B$ ← The PVM is performed only on $\mathcal{H}_B$.

It's a PVM!

  • $U_{AB}(|\cdot\rangle \otimes |0\rangle) = \sum_{i=1}^{N} (V_i \sqrt{E_i}) \otimes \mathbb{1} \, (|\cdot\rangle \otimes |i\rangle)$

where V_i are unitaries that we can choose.

• It is unitary since orthonormal vector are sent to orthonormal vector.
• We're defining it only on the subspace $|\psi\rangle \otimes |0\rangle$ $\forall |\psi\rangle$.

PROOF:

We want to show: $tr\left(\Pi_K \left(U_{AB}\, \rho \otimes |0\rangle_B \langle 0|_B\, U_{AB}^{\dagger}\right)\right) = tr(E_K \rho)$

Let's write ρ in the eigendec. $\Rightarrow \rho = \sum_{i=1}^{d} \lambda_i |\psi_i\rangle\langle\psi_i|$

$$tr\left(\Pi_K\left(U_{AB}\, \rho \otimes |0\rangle_B\langle 0|_B\, U_{AB}^\dagger\right)\right) = \sum_{i=1}^{d} \lambda_i\, tr\left(\Pi_K\, U_{AB}\, |\psi_i\rangle\langle\psi_i| \otimes |0\rangle\langle 0|\, U_{AB}^\dagger\right) =$$

$$= \sum_{i=1}^{d} \lambda_i\, tr\left(\Pi_K\, U_{AB}\left(|\psi_i\rangle \otimes |0\rangle\right)\left(\langle\psi_i| \otimes \langle 0|\right) U_{AB}^\dagger\right) =$$

$$= \sum_{i=1}^{d} \lambda_i\, tr\left(\Pi_K \sum_{j=1}^{N} \left(V_j \sqrt{E_j}\right) \otimes \mathbb{1} \left(|\psi_i\rangle \otimes |j\rangle\right) \left(\langle\psi_i| \otimes \langle 0|\right) U_{AB}^\dagger\right) =$$

$$U_{AB}\left(|\cdot\rangle \otimes |0\rangle\right) = \sum_{j=1}^{N} \left(V_j \sqrt{E_j}\right) \otimes \mathbb{1} \left(|\cdot\rangle \otimes |j\rangle\right)$$

$$= \sum_{i=1}^{d} \lambda_i\, tr\left(\Pi_K \sum_{j=1}^{N} \left(V_j \sqrt{E_j}\right) \otimes \mathbb{1} \left(|\psi_i\rangle \otimes |j\rangle\right) \sum_{l=1}^{N} \left(\langle\psi_i| \otimes \langle l|\right) \left(V_l \sqrt{E_l}\right)^\dagger \otimes \mathbb{1}\right)$$

$$= \sum_{i=1}^{d} \lambda_i\, tr\left(\mathbb{1} \otimes |k\rangle\langle k| \sum_{j=1}^{N} \left(V_j \sqrt{E_j}\right) \otimes \mathbb{1} \left(|\psi_i\rangle \otimes |j\rangle\right) \sum_{l=1}^{N} \left(\langle\psi_i| \otimes \langle l|\right) \left(V_l \sqrt{E_l}\right)^\dagger \otimes \mathbb{1}\right)$$

$$\Pi_K = \mathbb{1} \otimes |k\rangle\langle k|$$

$$= \sum_{i=1}^{d} \lambda_i\, tr_{AB}\left(\mathbb{1} \otimes |k\rangle\langle k| \left(V_K \sqrt{E}\right) \otimes \mathbb{1} \left(|\psi_i\rangle \otimes |k\rangle\right) \left(\langle\psi_i| \otimes \langle k|\right) \left(V_K \sqrt{E_K}\right)^\dagger \otimes \mathbb{1}\right)$$

TAKING TRACE RESPECT TO B

$$= \sum_{i=1}^{d} \lambda_i\, tr_A\left(\left(V_K \sqrt{E_k}\right) |\psi_i\rangle\langle\psi_i| \left(V_K \sqrt{E_K}\right)^\dagger\right) =$$

$$= tr\left(\left(V_K \sqrt{E_K}\right)^\dagger V_K \sqrt{E_K}\, \rho\right) = tr\left(\sqrt{E_K}\sqrt{E_K}\,\rho\right) = tr\left(E_K\, \rho\right)$$

- $\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$

- $V_K V_K^\dagger = \mathbb{1}$
- $\left(\sqrt{E_K}\right)^\dagger = \sqrt{E_K}$

$E_K \geq 0 \Rightarrow \sqrt{E_K} \geq 0 \Rightarrow \left(\sqrt{E_K}\right)^\dagger = \sqrt{E_K}$

$\Rightarrow \square$

- If the POVM is actually implemented physically by a PVM in a larger system, then the post measurement state would be:

AXIOMS PVM.

$$\rho_{\substack{\text{AFTER}\\ \text{"k" OUTCOME}}} = \frac{\Pi_k \, \rho' \, \Pi_k}{tr(\rho' \Pi_k)} \qquad \text{where } \rho' = U_{AB}\, \rho \otimes |0\rangle_B \langle 0|_B \, U_{AB}^\dagger .$$

- Using our construction with $\Pi_k = \mathbb{1} \otimes |k\rangle\langle k|$ and U_{AB} we have:

$$\rho^{(AB)}_{\substack{\text{AFTER}\\ \text{"k" OUTCOME}}} = \frac{\Pi_k \, \rho' \, \Pi_k}{tr(\rho' \Pi_k)} \underset{\text{BEFORE}}{=} \frac{\Pi_k \, \rho' \, \Pi_k}{tr(\rho E_k)} = \left(V_k \, \frac{\sqrt{E_k}\, \rho \sqrt{E_k}}{tr(\rho E_k)} \, V_k^\dagger \right) \otimes |k\rangle\langle k|$$

- $U_{AB} \left(|\cdot\rangle \otimes |0\rangle \right) = \sum_{j=1}^{N} \left(V_j \sqrt{E_j} \right) \otimes \mathbb{1} \left(|\cdot\rangle \otimes |j\rangle \right)$
- $\rho' = U_{AB}\, \rho \otimes |0\rangle_B \langle 0|_B \, U_{AB}^\dagger$
- $\Pi_k = \mathbb{1} \otimes |k\rangle\langle k|$
- SIMILARLY AS BEFORE

tracing out the system "B".

$$\Rightarrow \quad \boxed{\rho_{\substack{\text{AFTER}\\ \text{"k" OUTCOME}\\ \text{POVM}}} = V_k \, \frac{\sqrt{E_k}\, \rho \sqrt{E_k}}{tr(\rho E_k)} \, V_k^\dagger}$$

$\Leftarrow$ POST POVM OUTCOME "k" STATE

"we need to know V_k which depend by the "physical" implementation of the PVM."