Random circuits under noise

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Most circuits under noise

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Noise-induced shallow circuits and absence of barren plateaus

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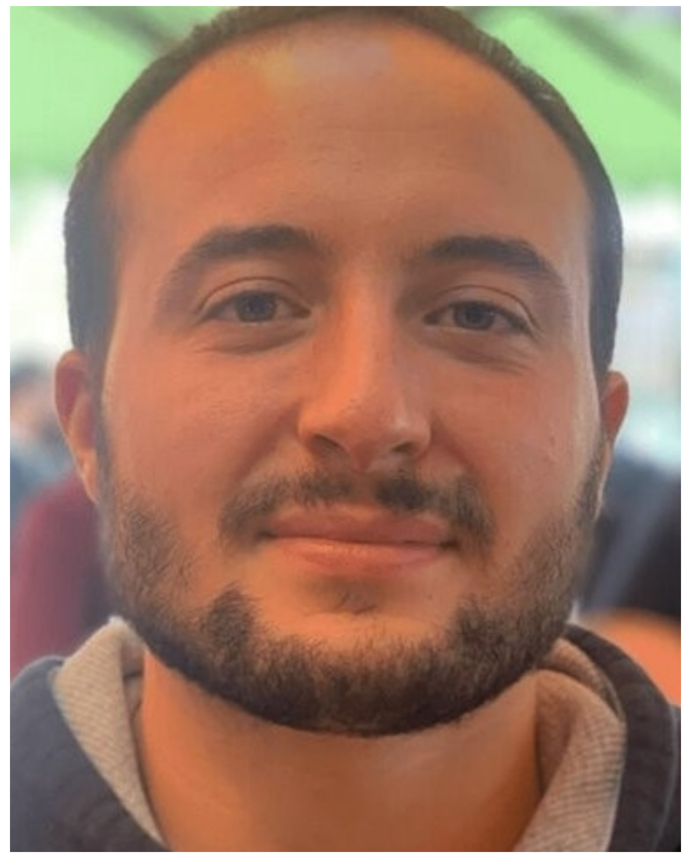
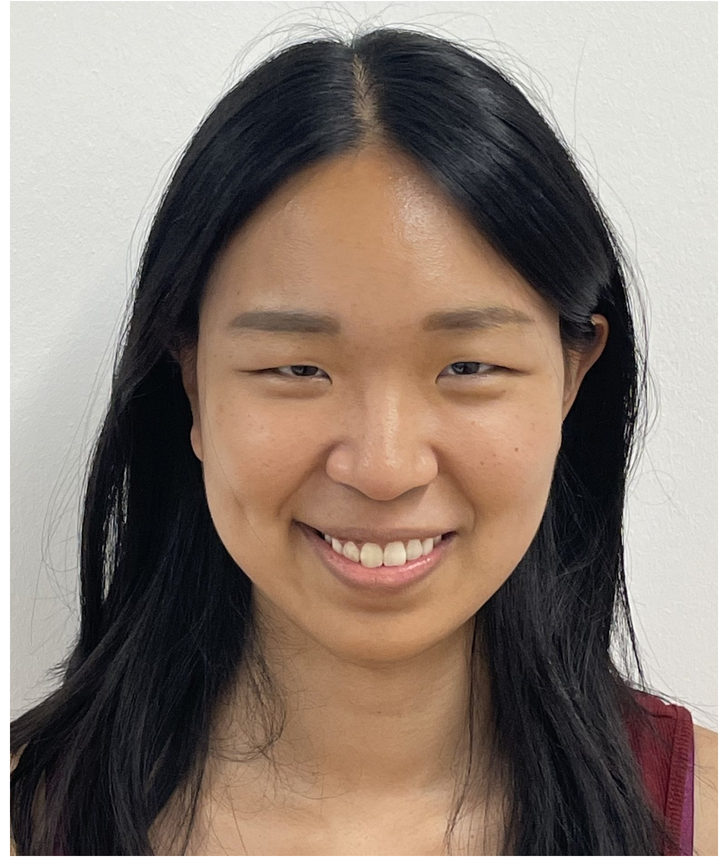
Joint work with:

Armando Angrisani, Soumik Ghosh, Sumeet Khatri, Jens Eisert, Daniel Stilck França, Yihui Quek

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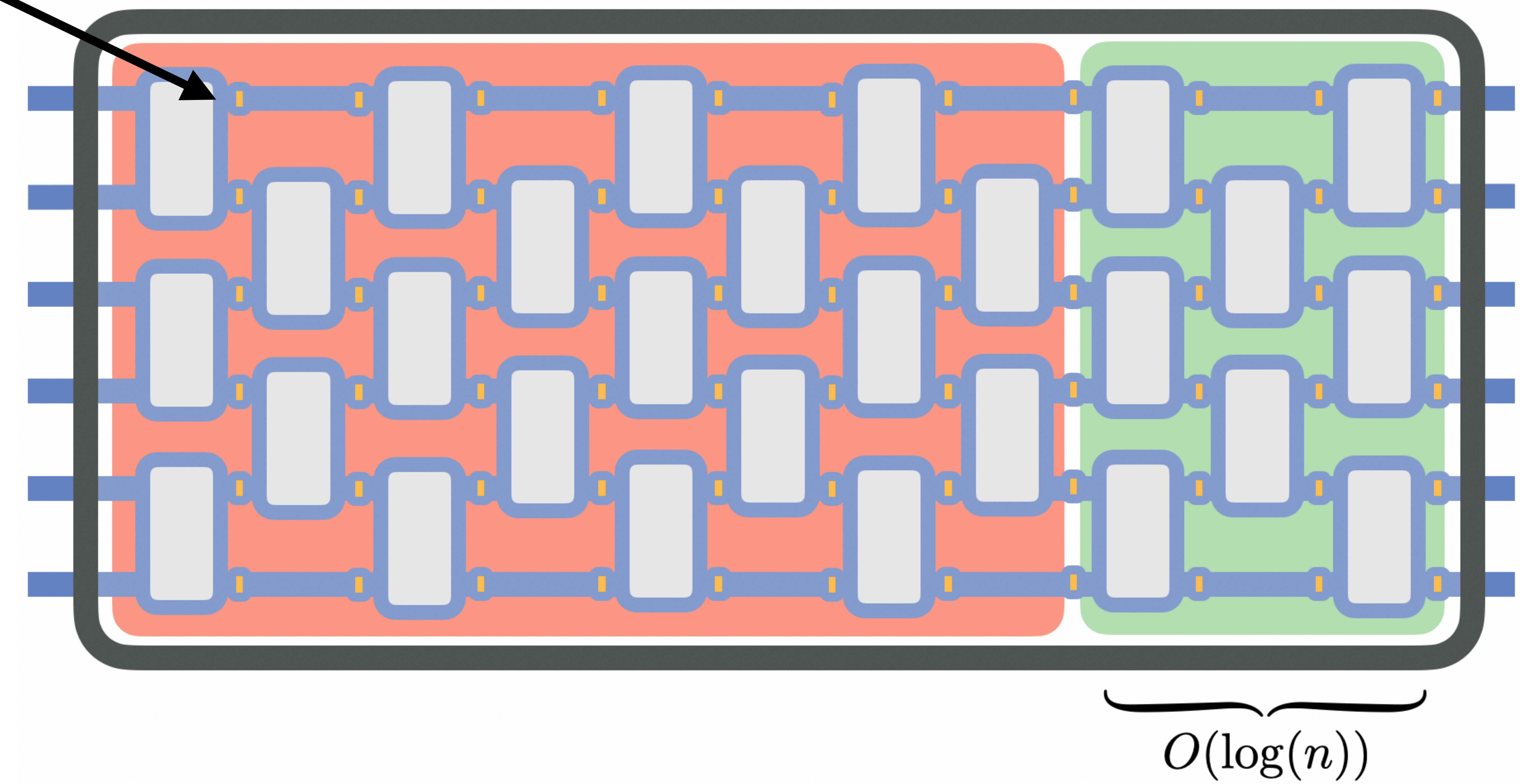
Spoiler

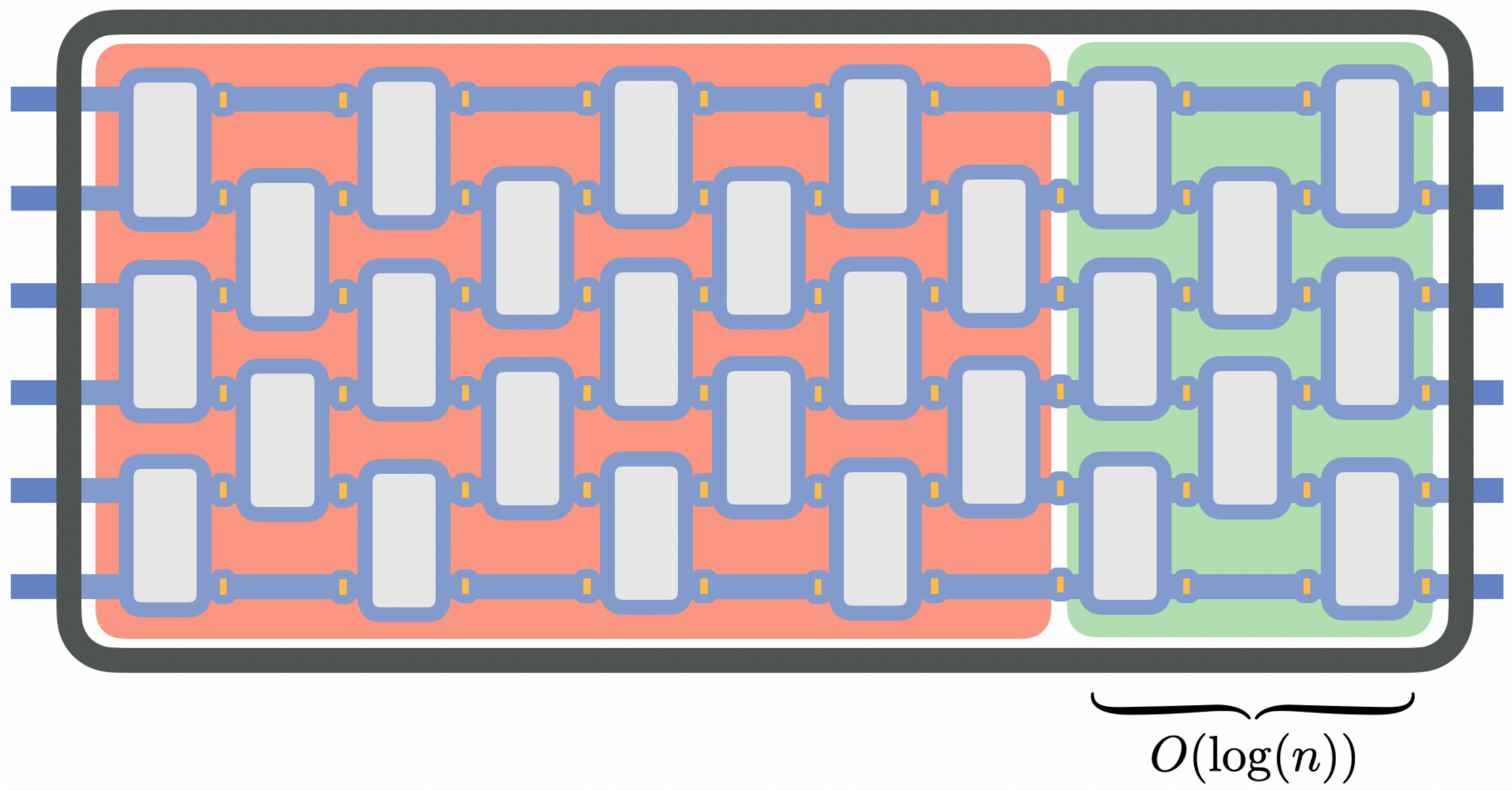
(What we'll see)



Only the last layers matter to estimate expectation values, for most circuits.

This implies:
- Efficient classical simulation
- Lack of "Barren plateaus"

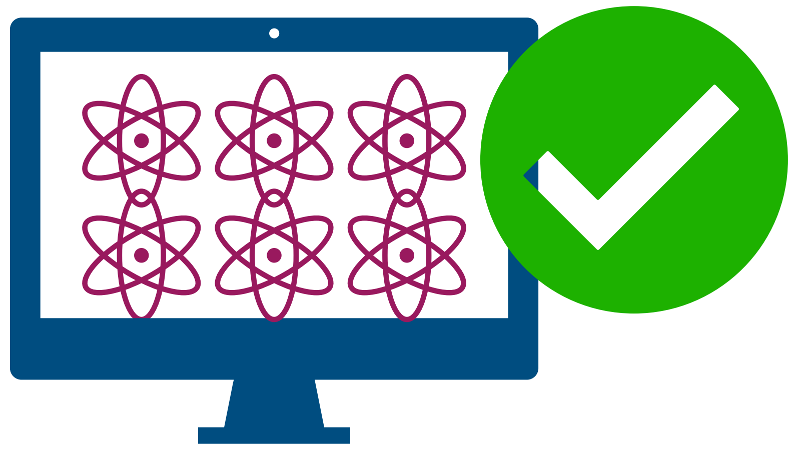
Outline



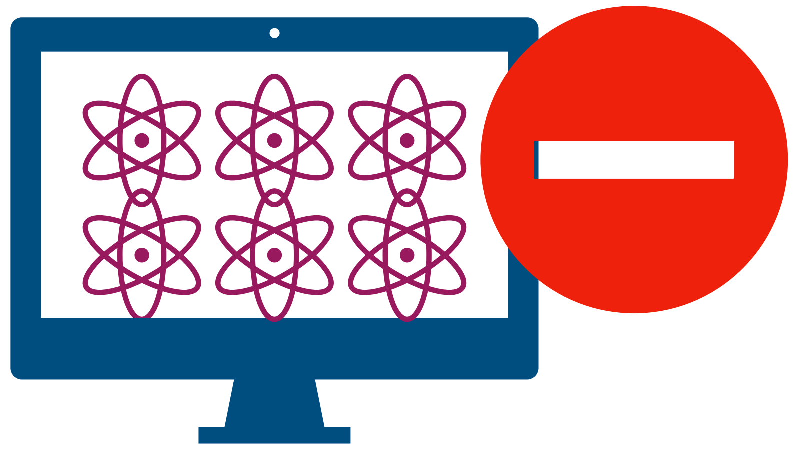
- Noise in quantum circuits
- Effective shallow circuits
- Classical simulation of Pauli expectation values of noisy random circuits
- Barren plateaus

- Understanding **noise impact** in current quantum devices is important for:

Capabilities



Limitations



- Many previous works modeled noise as solely **depolarizing**.

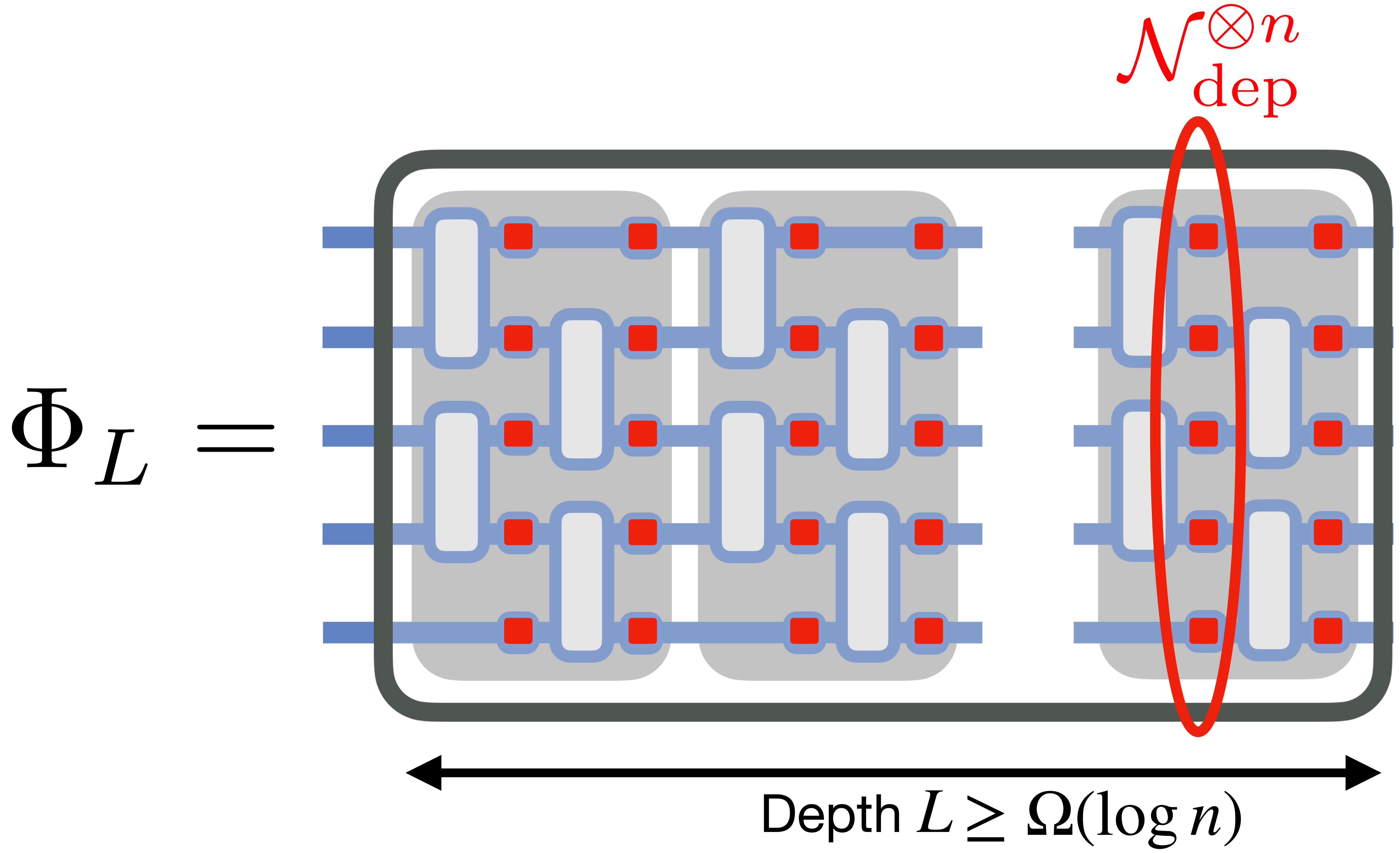
$$\mathcal{N}(\sigma) = (1-p)\sigma + p\frac{I}{2} \quad \text{with } p \in [0,1]$$

- **Depolarizing noise**, e.g., induces:
  - Barren plateaus in variational quantum algorithms [1],
  - Efficient classical simulation of **'supremacy' sampling experiments** [2].

- But, **tiny departure** from this model could lead to **different conclusions** [3,4].

[1] Noise-induced barren plateaus in variational quantum algorithms. Wang *et al,* Nature Comm. *(2021).*
[2] A polynomial-time classical algorithm for noisy random circuit sampling, Aharonov *et al*., STOC (2023)
[3] Quantum refrigerator, Ben-Or *et al*, QIP (2013).
[4] Effect of non-unital noise on random circuit sampling, Fefferman *et al*, QIP (2023).

- The **depolarizing** assumption, if true, **enormously constrains noisy computation**:



$$\left\| \Phi_L(\rho_0) - \frac{\mathbb{I}}{2^n} \right\|_1 \leq \exp(-\Theta(L)),$$ [5]

$\Downarrow$

Quantum computation possible only for $\log(n)$ depth! [6]

$\Downarrow$

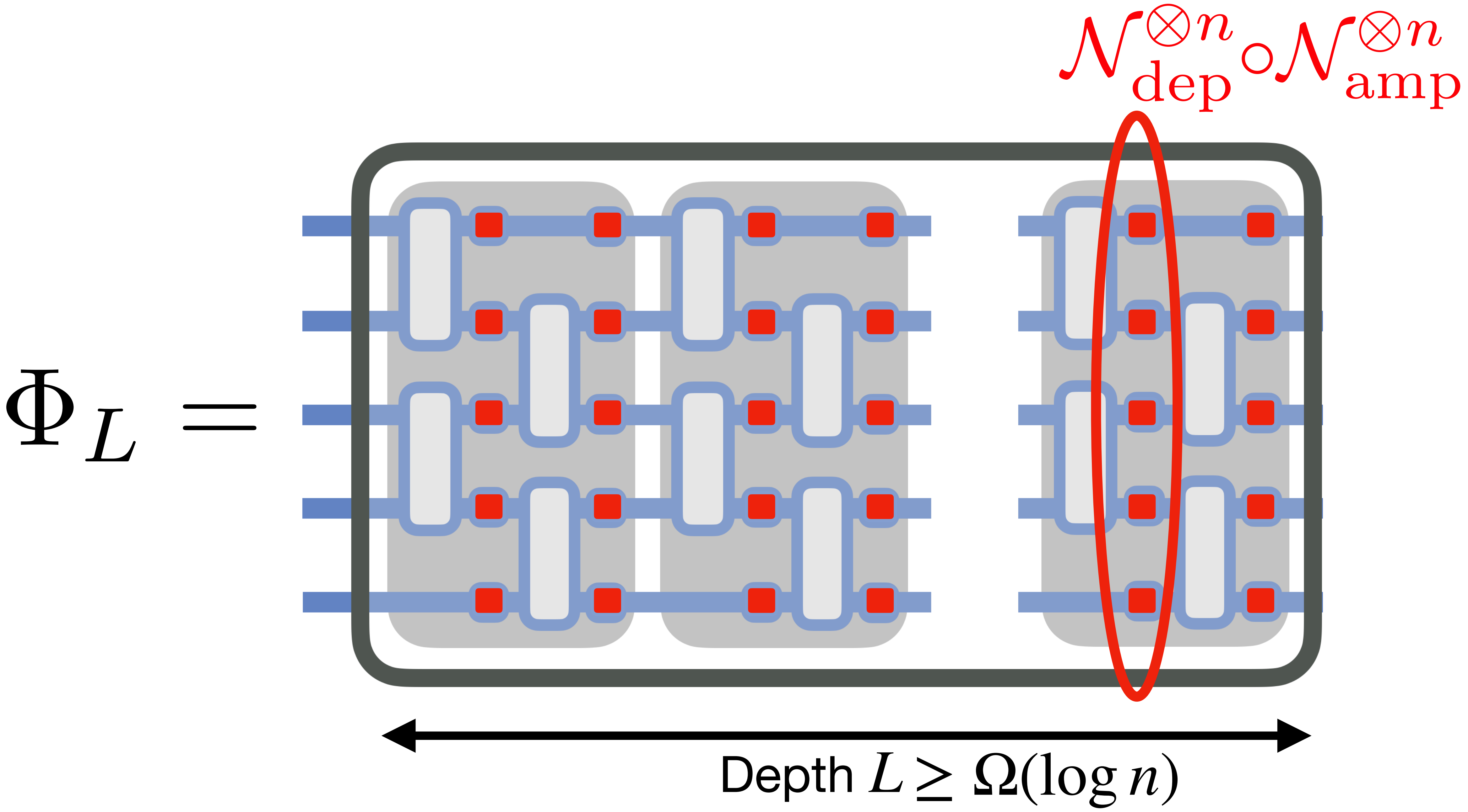
$$|\operatorname{Tr}(P\Phi_L(\rho_0))| \leq \exp(-\Theta(L)),$$

for any Pauli P and input state ρ_0

- Thus, **efficient** classical simulation of **all** deep depolarizing-noisy circuits is "trivial".

[5] Relative entropy convergence for depolarizing channels, Muller-Hermes *et al,* J. Math. Phys. (2016).
[6] Limitations of noisy reversible computation, Aharonov et al, ArXiv (1996).

- The **depolarizing** assumption, if true, **enormously constrains noisy computation**:



$$\left\| \Phi_L(\rho_0) - \frac{\mathbb{I}}{2^n} \right\|_1 \leq \exp(-\Theta(L)),$$ [5]

Quantum computation possible only for $\log(n)$ depth! [6]

However, for noises different from depolarizing, not all circuits become trivial after log-depth!

- Real **hardware noise** is **not solely depolarizing** (which is 'unital', i.e., $\mathcal{N}(I) = I$)
- Other noise components present (T1-error, amplitude damping, …) are **non-unital**.

[3] Quantum refrigerator, Ben-Or *et al*, QIP (2013). → **'Non-unital' noisy circuits can be made fault-tolerant, without fresh auxiliary qubits!**

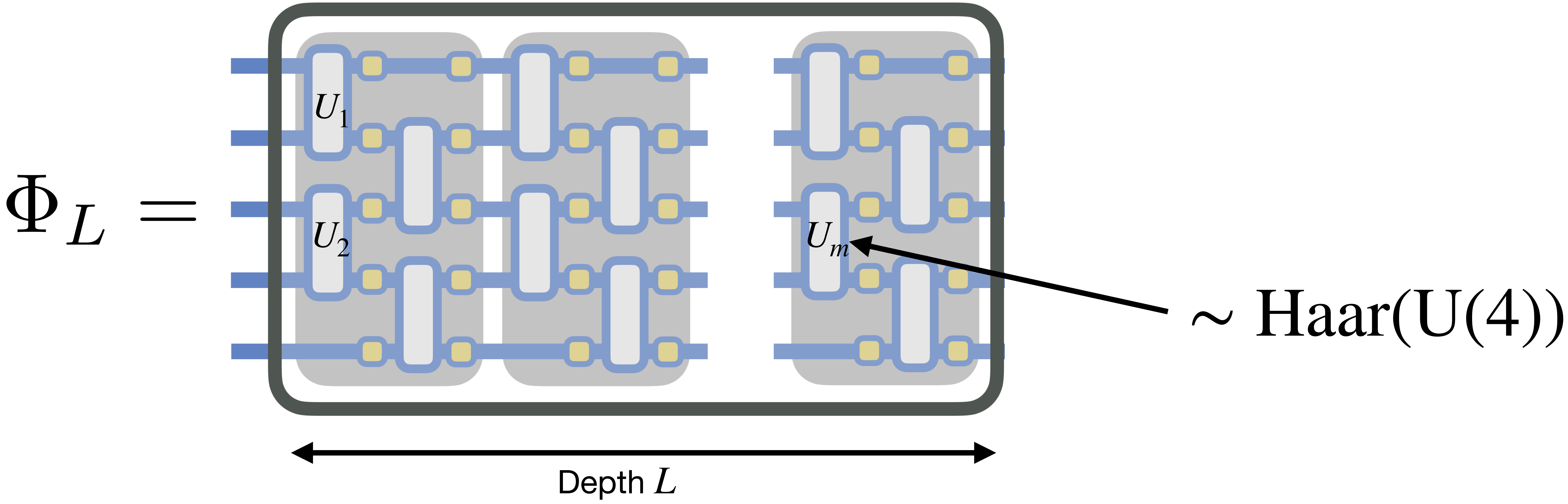
[5] Relative entropy convergence for depolarizing channels, Muller-Hermes *et al,* J. Math. Phys. (2016).
[6] Limitations of noisy reversible computation, Aharonov et al, ArXiv (1996).

Thus, if the noise is non-unital:
one can carefully engineer the circuit to perform reliable quantum computation [3]

(If BQP≠BPP, Pauli expectation values of **all** non-unital noisy circuits cannot be efficiently estimated classically.)

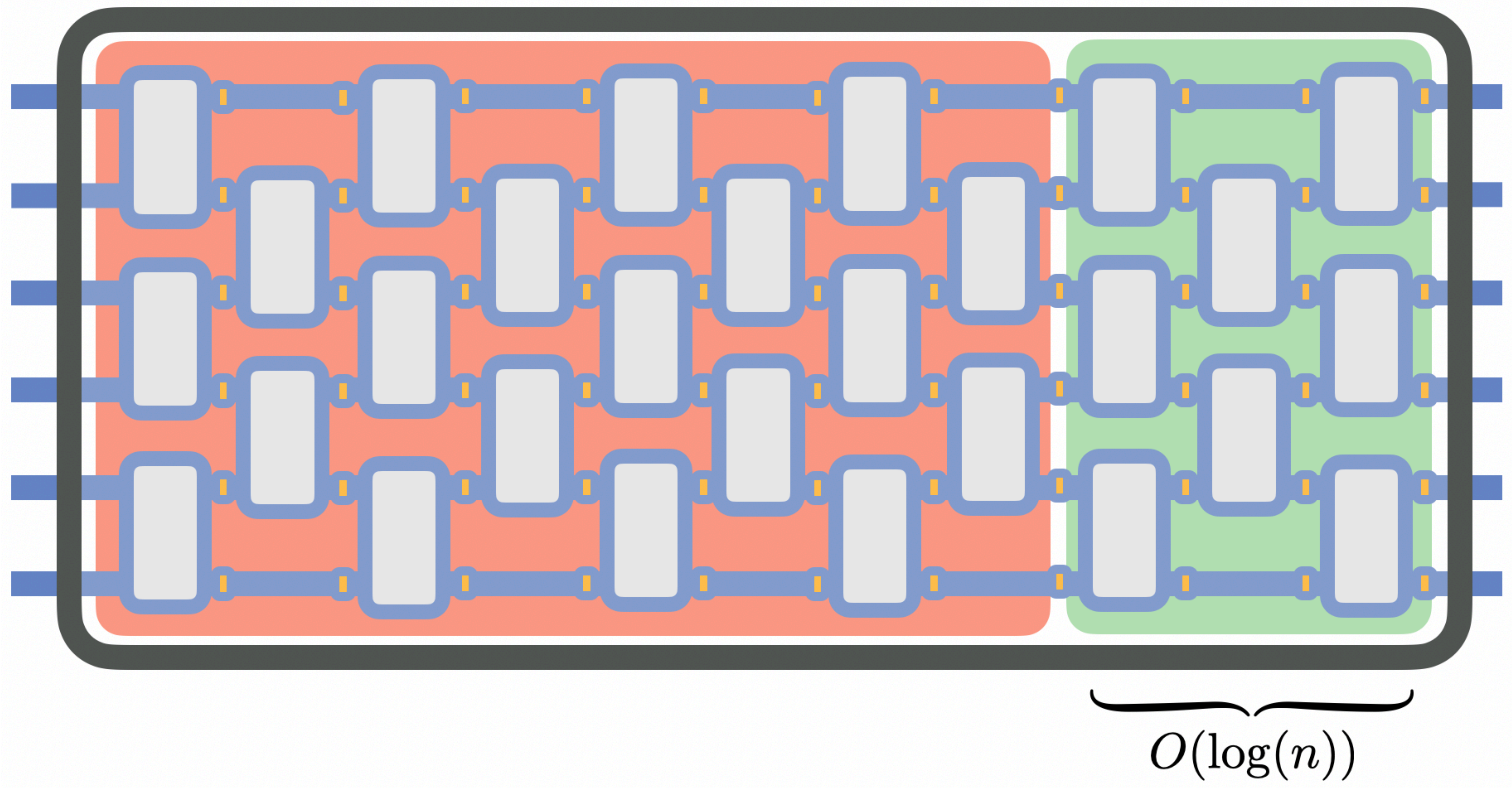
But, what happens for generic noisy circuits?

(Our work)



[3] Quantum refrigerator, Ben-Or *et al*, QIP (2013).

What happens for generic noisy circuits?

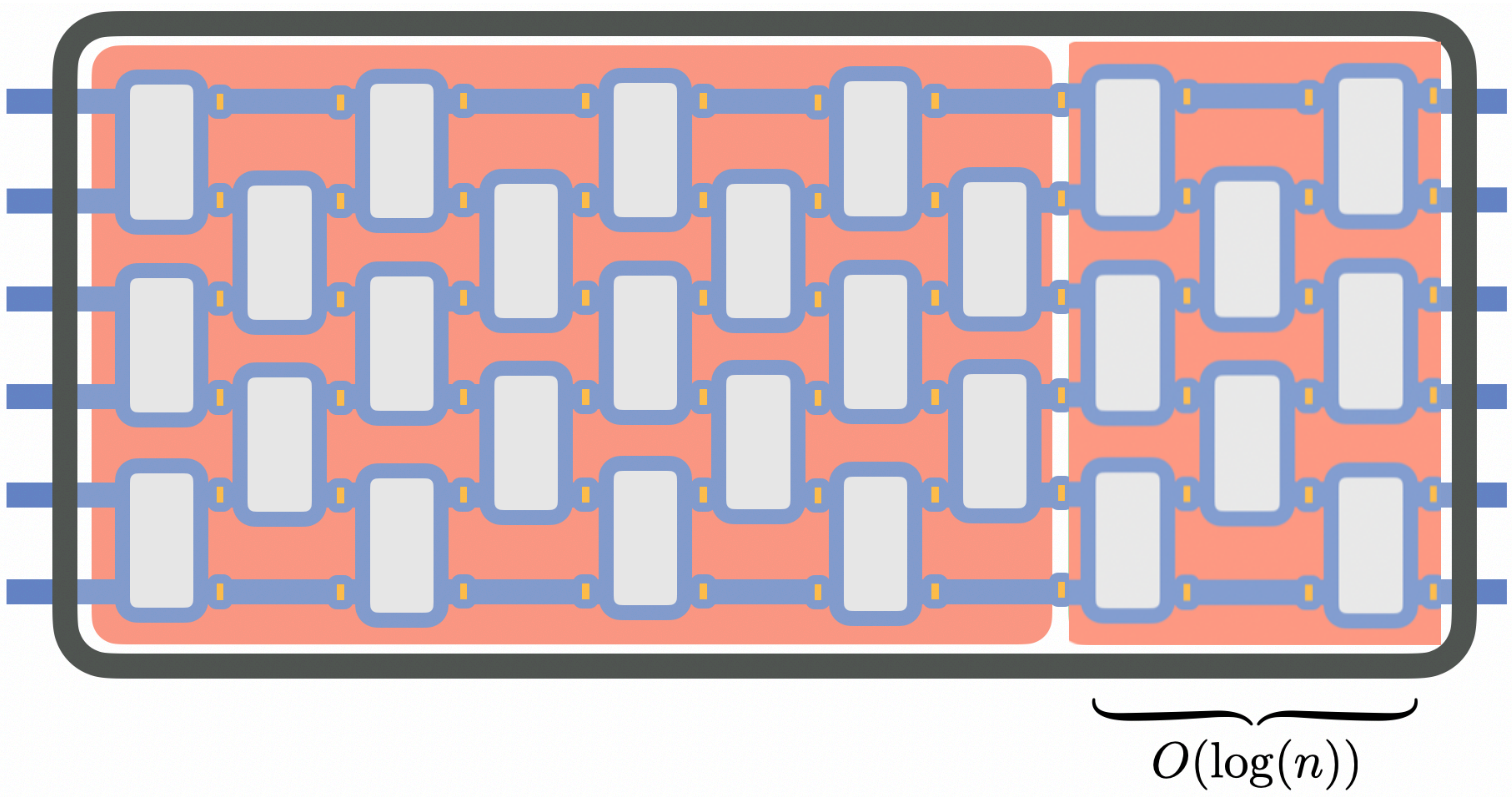


Only the last $\log(n)$ layers matter to estimate expectation values, for most circuits.

Intuition: the content of the first layers is forgotten, if nothing *clever* is done to remember!

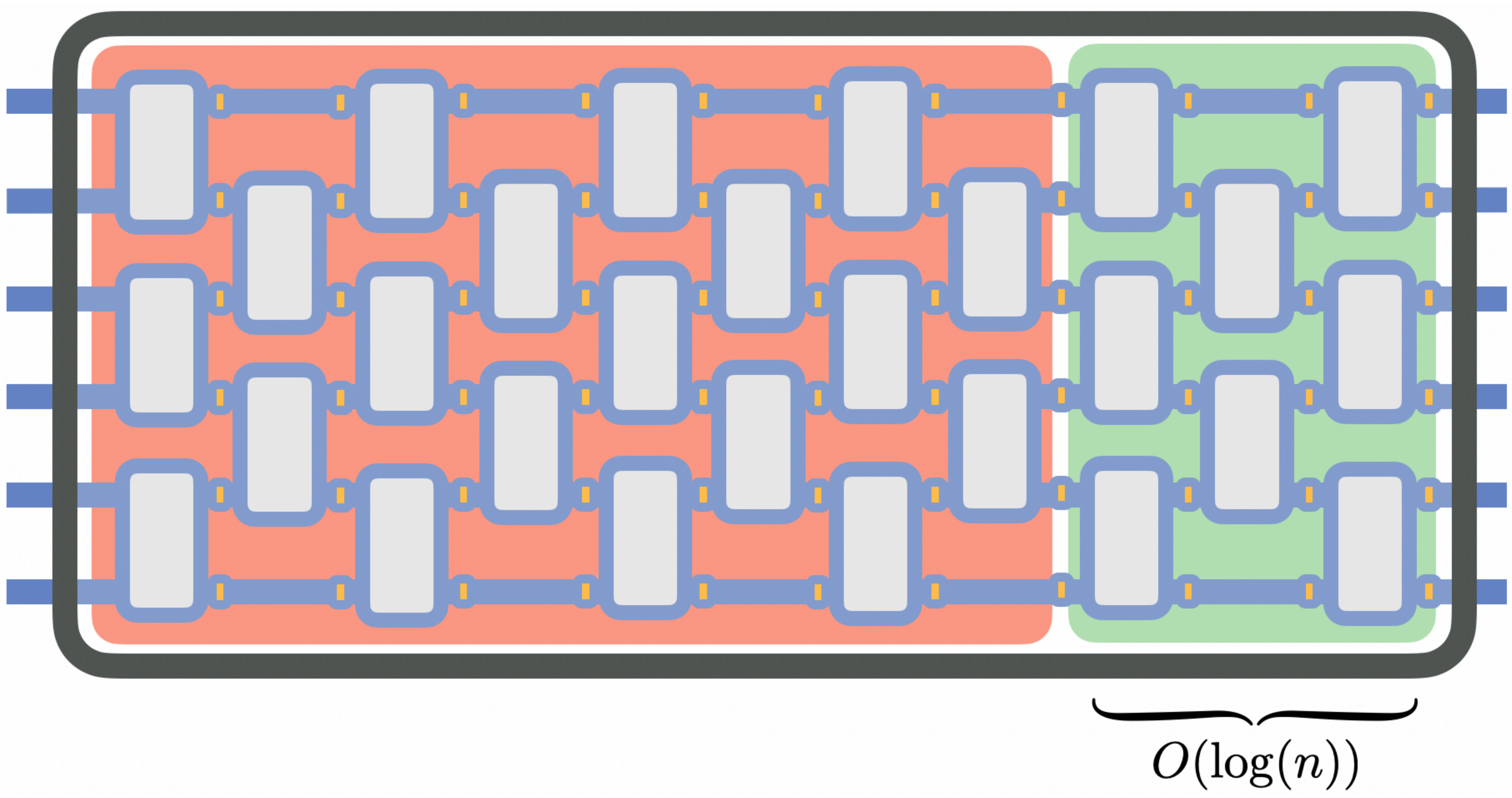
If the noise is depolarizing, no layers matter.

(Even for ALL circuits)



If the noise is **non-unital**, the last $O(\log(n))$ do matter!

(for most circuits)



Effective shallow circuits

Proposition.

For any initial state ρ_0 (possibly complex) and observable O,

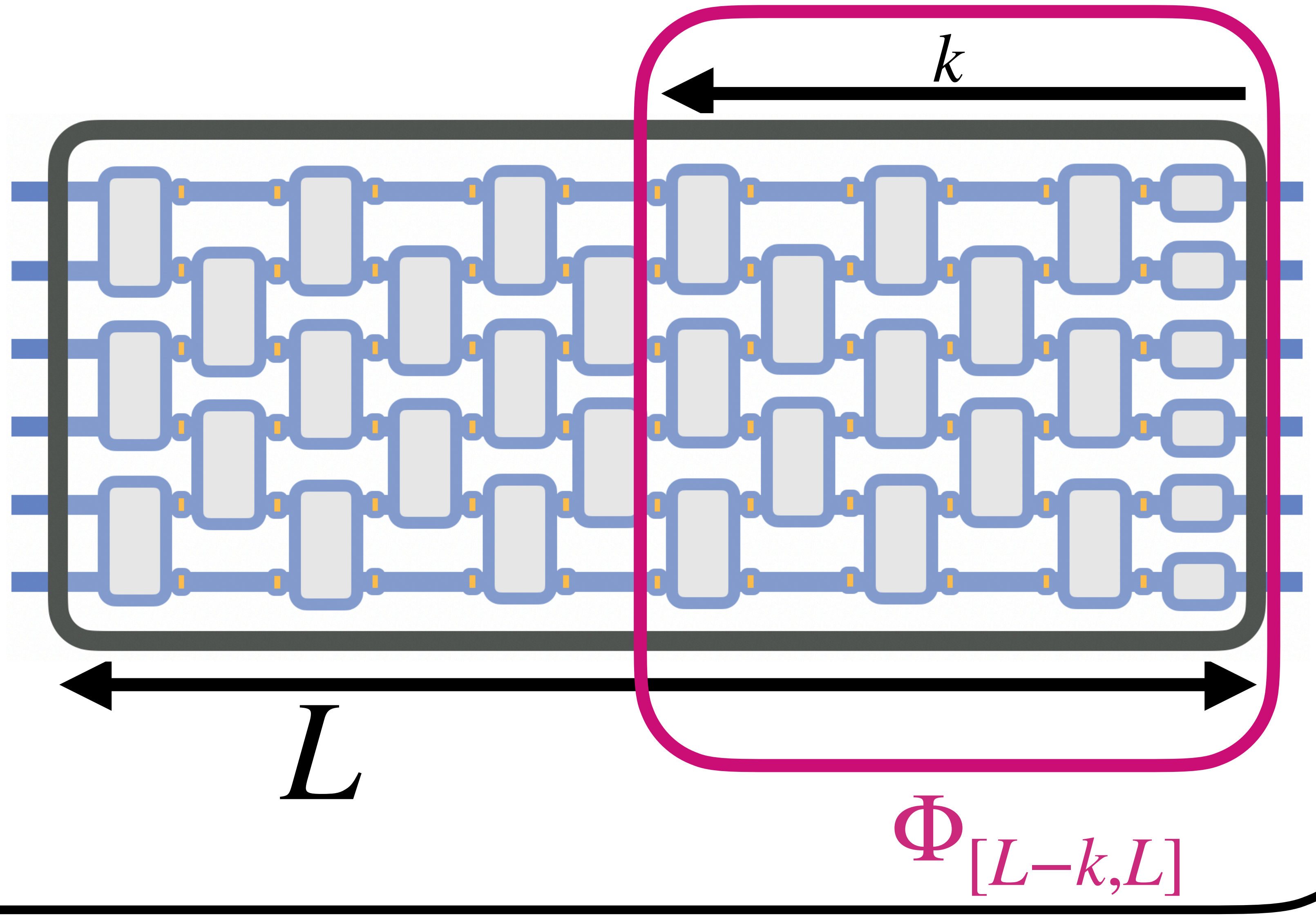
$$\mathbb{E}_{\Phi}\left[\left|\mathrm{Tr}\left(O\Phi(\rho_0)\right) - \mathrm{Tr}\left(O\Phi_{[L-k,L]}(\sigma_0)\right)\right|\right] \leq \|O\|_\infty \exp(-\Omega(k)).$$

Target expectation value

Last k layers

Any preferred initial state

$\Phi =$



This follows from the following more general theorem:

Theorem.

For any k depth of the circuit Φ_k, states ρ, σ, and Pauli P, we have:

$$\mathbb{E}_{\Phi_k}\left[\left|\mathrm{Tr}(P\Phi_k(\rho)) - \mathrm{Tr}(P\Phi_k(\sigma))\right|\right] \leq \exp(-\Omega(k + |P|)).$$

Trace distance convergence

Theorem. (Trace distance bound)

For any $L \geq \Omega(n)$ depth of the noisy n-qubits circuit Φ, states ρ, σ, we have:

$$\mathbb{E}_{\Phi} \|\Phi(\rho) - \Phi(\sigma)\|_1 \leq \exp(-\Omega(L)).$$

- A general result like this, but without $\mathbb{E}_{\Phi}$, cannot be obtained (e.g.) because of Ref. [3]

  (which shows how to leverage non-unital noise to perform fault-tolerant quantum computation without fresh-auxiliary qubits.)

- However, in certain 'high noise' regime, we can obtain it:

Theorem. (High-noise worst-case, informal)

For any $L \geq \Omega(\log(n))$ depth of the (very) noisy n-qubits circuit Φ, states ρ, σ, we have:

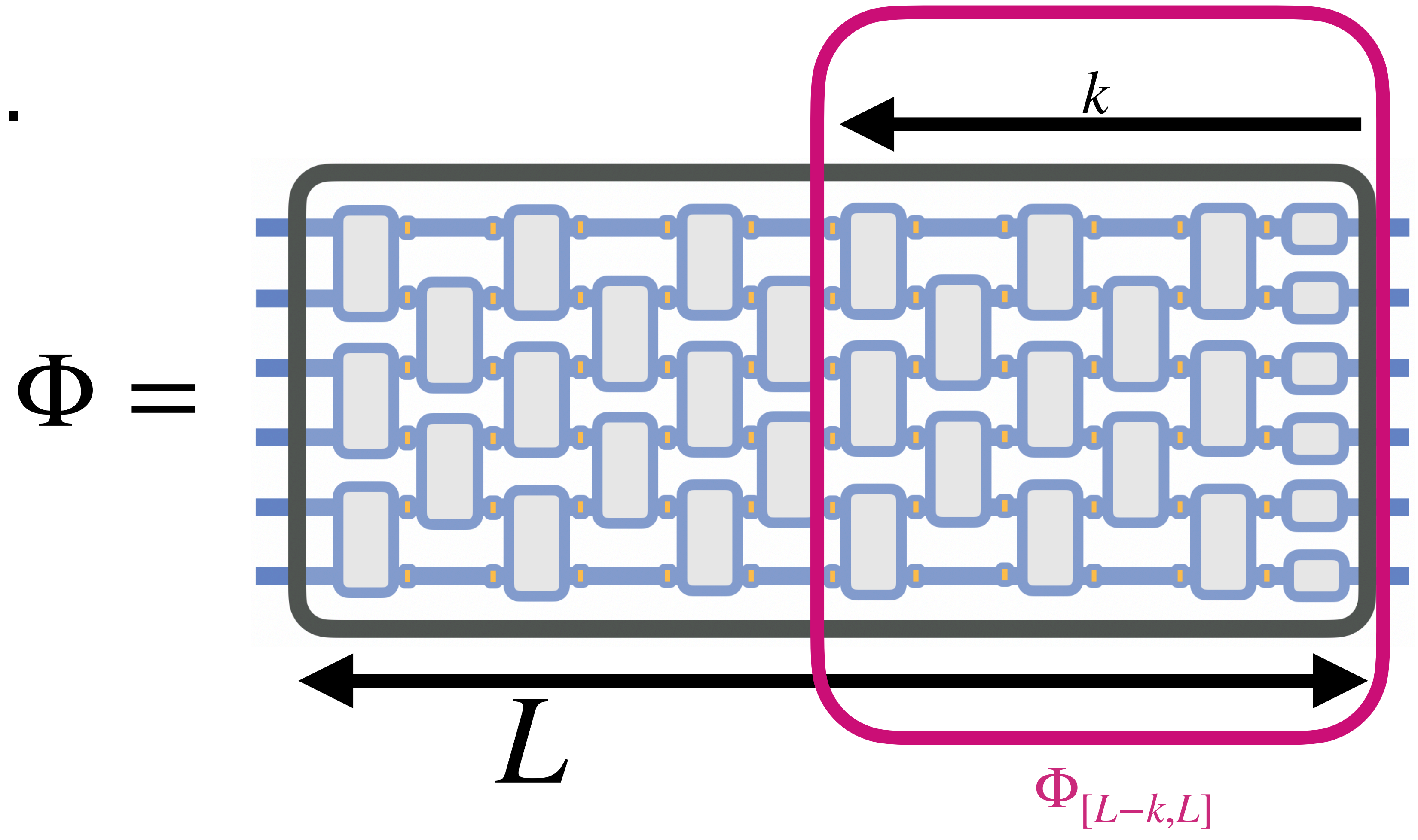
$$\|\Phi(\rho) - \Phi(\sigma)\|_1 \leq \exp(-\Omega(L)).$$

[3] Quantum refrigerator, Ben-Or *et al*, QIP (2013).

Classical simulation of noisy random circuits

Task: Estimate $\mathrm{Tr}(P\Phi(\rho_0))$, with high probability over the choice of Φ.

Solution: It suffices to output $\mathrm{Tr}\left(P\Phi_{[L-k,L]}(|0^n\rangle\langle 0^n|)\right)$.



Previous proposition.

$$\mathbb{E}_\Phi \left[\left|\mathrm{Tr}\left(P\Phi(\rho_0)\right) - \mathrm{Tr}\left(P\Phi_{[L-k,L]}(|0^n\rangle\langle 0^n|)\right)\right|\right] \leq \exp(-\Omega(k+|P|)).$$

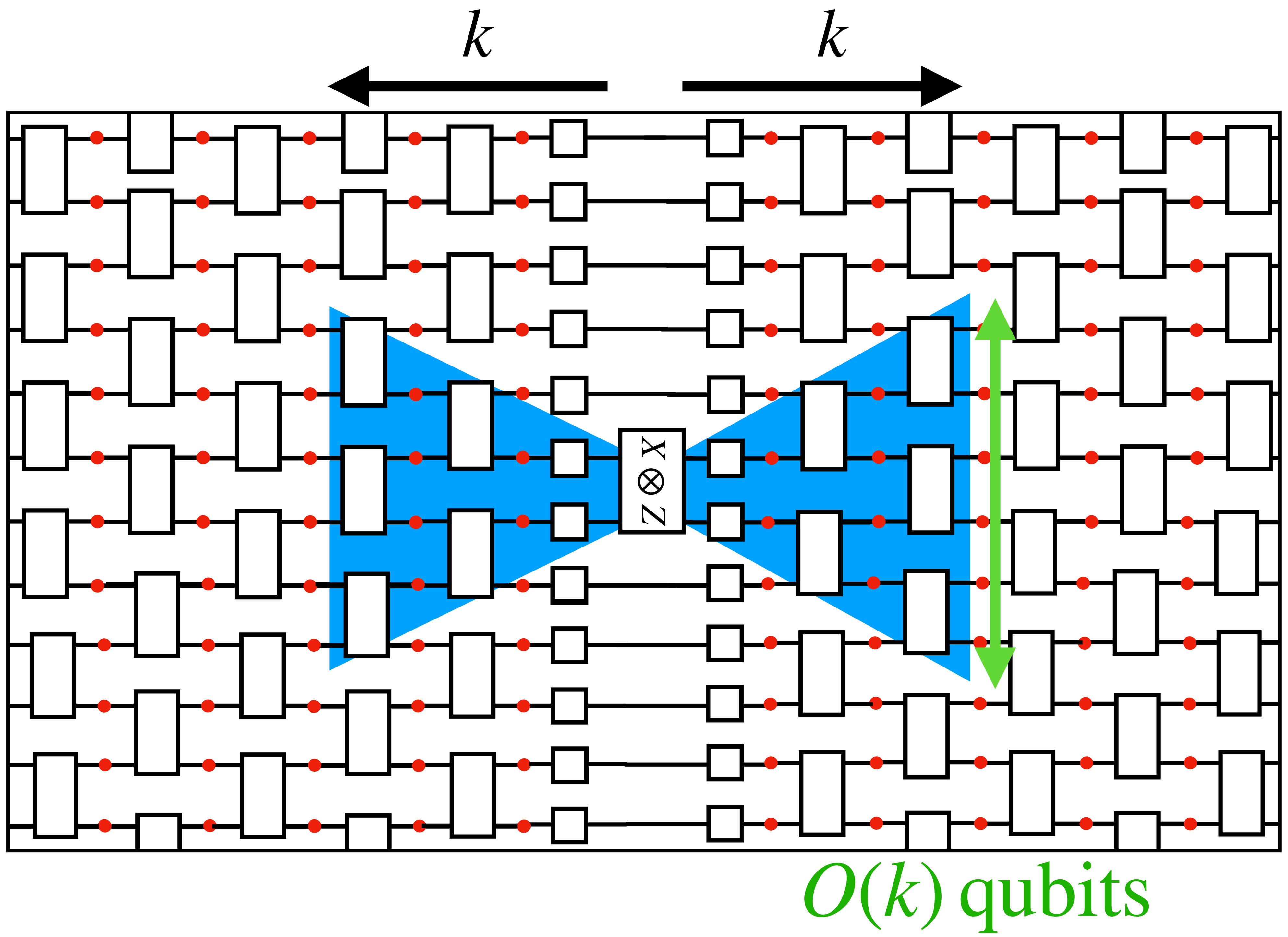
Choosing $k = O(\log(\varepsilon^{-1}))$ suffices to have: $\leq \varepsilon$

- We have $\mathrm{Tr}\left(P\Phi_{[L-k,L]}(|0^n\rangle\langle 0^n|)\right) = \mathrm{Tr}\left(\Phi_{[L-k,L]}{}^*(P)|0^n\rangle\langle 0^n|\right)$

If P is local, then this is also local

- Thus, this can be computed efficiently for light-cone arguments.

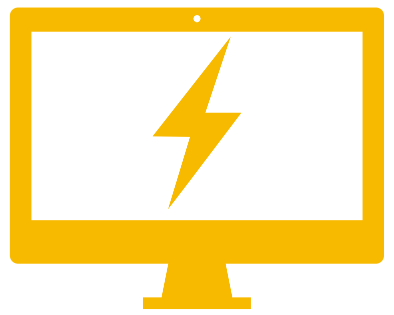
Computational time (for 1D): $\exp(O(k)) = \mathrm{poly}(\varepsilon^{-1})$



Classical simulation of noisy random quantum circuits expectation values

Theorem.

Given a randomly sampled noisy quantum circuit, for any depth and any noise, there is a classical algorithm to **estimate local Pauli expectation values** up to ε additive precision, with high probability over the choice of the circuit.



The runtime is $\exp\big(O(\log^{\mathrm{D}}(\varepsilon^{-1}))\big)$, where D is the spatial dimension of the system.

(No dependence by the number of qubits and depth.)

- Thus, for constant ε precision, the algorithm is efficient for any dimensionality.

- For $\varepsilon = \mathrm{poly}(n^{-1})$, the algorithm is efficient for 1D architectures, and quasi-poly for $\mathrm{D} > 1$.

- For **global Pauli** expectation values, output zero succeeds with high probability.

- We also give a **condition to verify the success of the classical simulation**:
  "Check if the Heisenberg evolved observable is $\sim$ proportional to the Identity".

[Submitted on 20 Mar 2024 (v1), last revised 10 Oct 2024 (this version, v2)]

Noise-induced shallow circuits and absence of barren plateaus

Antonio Anna Mele, Armando Angrisani, Soumik Ghosh, Sumeet Khatri, Jens Eisert, Daniel Stilck França, Yihui Quek

- **OPEN QUESTION FROM THIS WORK:**
  Simulating Pauli expectation values of noisy random quantum circuits with $\mathrm{poly}(n^{-1})$ accuracy in 2D and all-to-all connectivity in polynomial time.

- We solved this open question in:

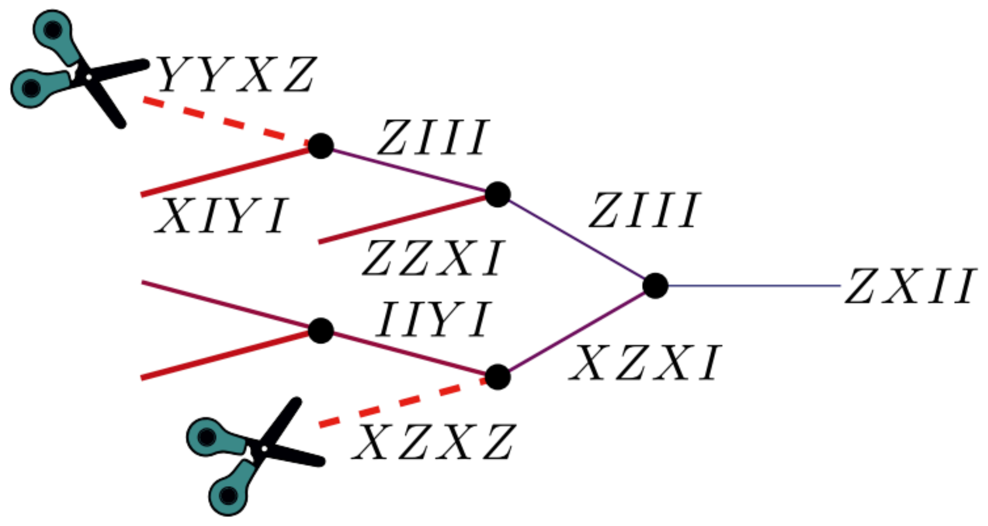
[Submitted on 22 Jan 2025]

Simulating quantum circuits with arbitrary local noise using Pauli Propagation

Armando Angrisani, Antonio A. Mele, Manuel S. Rudolph, M. Cerezo, Zoe Holmes

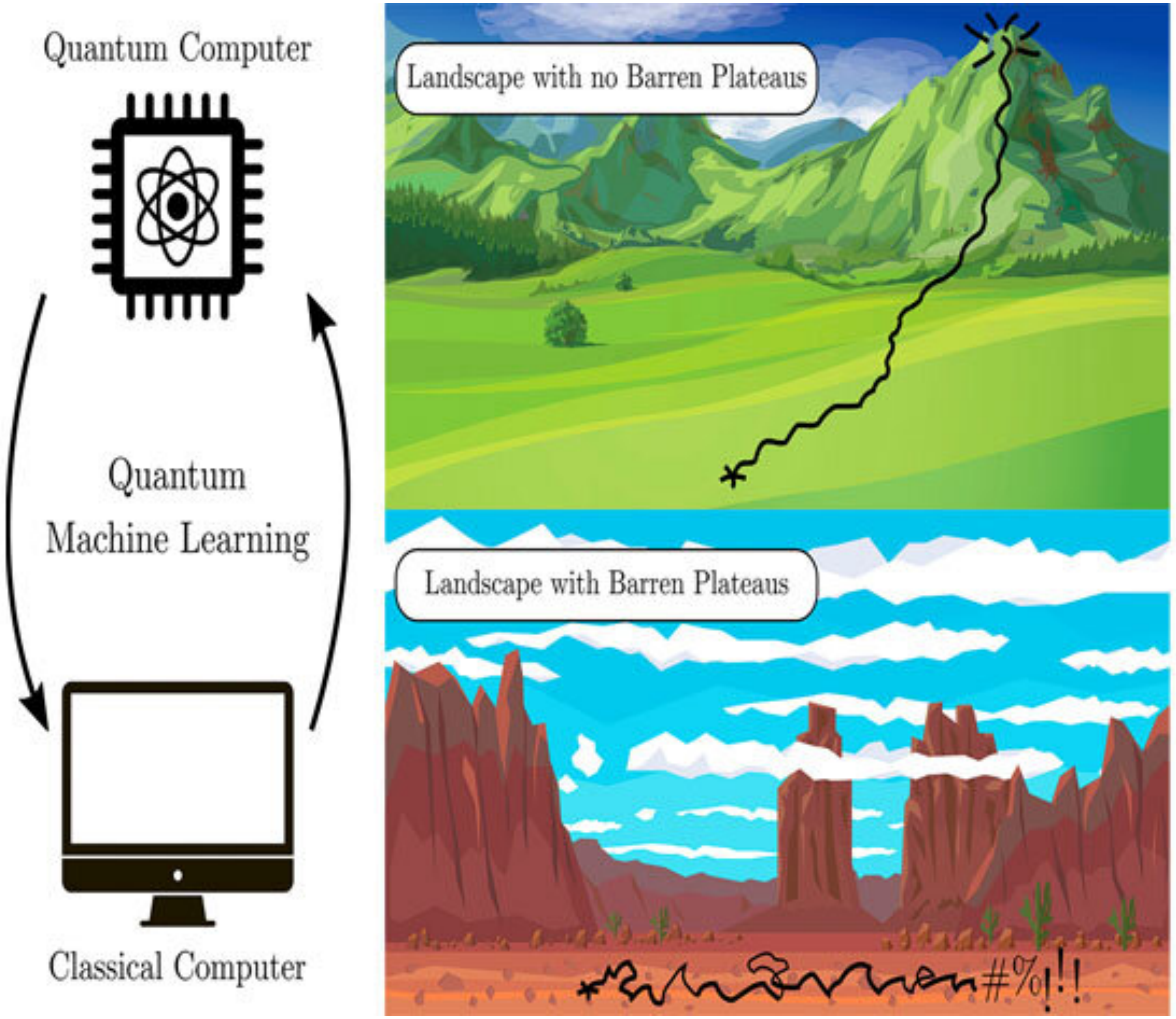
We present a polynomial-time classical algorithm for estimating expectation values of arbitrary observables on typical quantum circuits under any incoherent local noise, including non-unital or dephasing. Although previous research demonstrated that some carefully designed quantum circuits affected by non-unital noise cannot be efficiently simulated, we show that this does not apply to average-case circuits, as these can be efficiently simulated using Pauli-path methods. Specifically, we prove that, with high probability over the circuit gates choice, Pauli propagation algorithms with tailored truncation strategies achieve an inversely polynomially small simulation error. This result holds for arbitrary circuit topologies and for any local noise, under the assumption that the distribution of each circuit layer is invariant under single-qubit random gates. Under the same minimal assumptions, we also prove that most noisy circuits can be truncated to an effective logarithmic depth for the task of {estimating} expectation values of observables, thus generalizing prior results to a significantly broader class of circuit ensembles. We further numerically validate our algorithm with simulations on a 6×6 lattice of qubits under the effects of amplitude damping and dephasing noise, as well as real-time dynamics on an 11×11 lattice of qubits affected by amplitude damping.

- Using "Pauli-path truncation" algorithm.



Barren plateaus

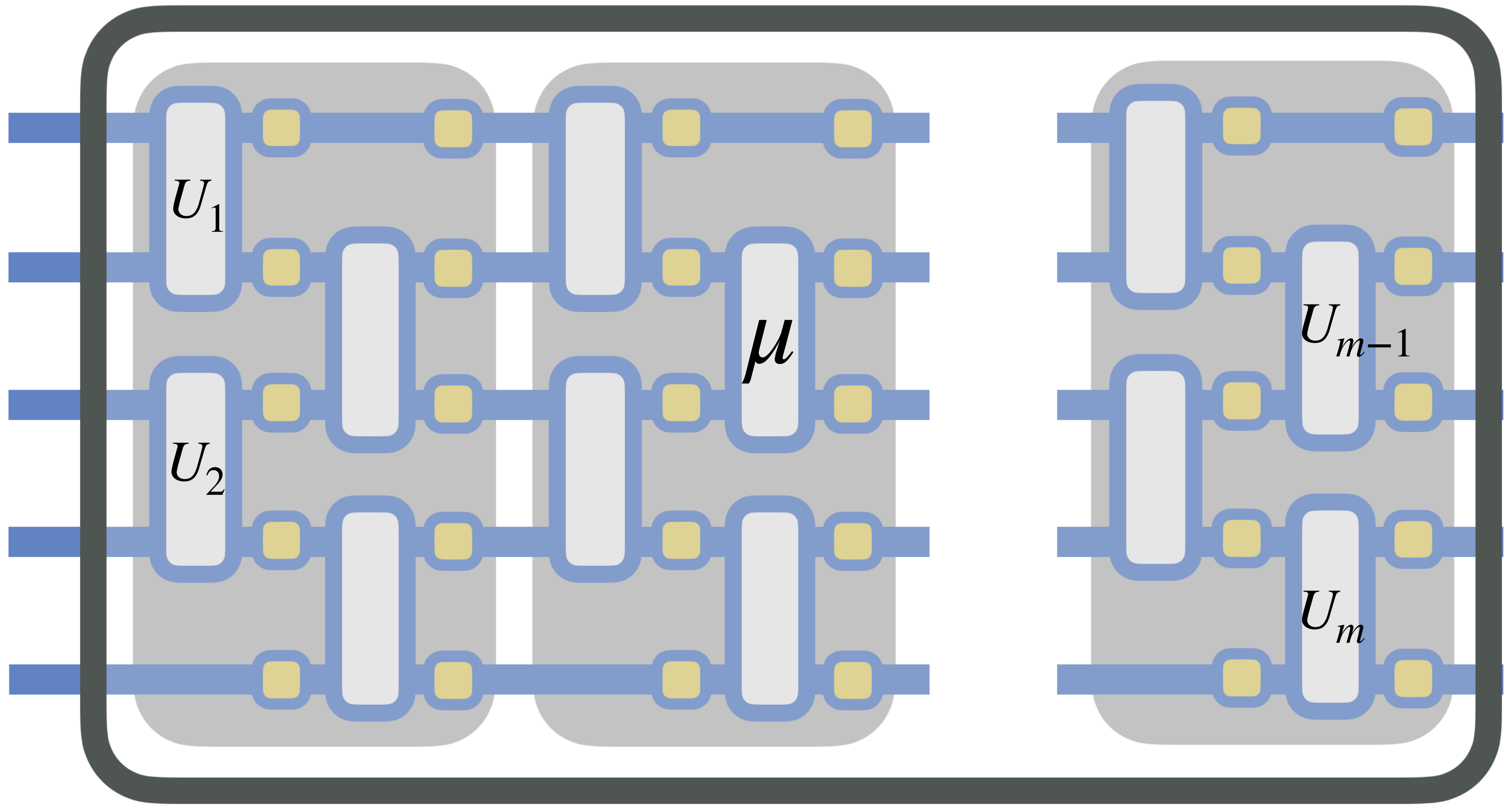
- **Variational quantum algorithms** encode the problem into $\min_{U_1,\ldots,U_m} \mathrm{Cost}(\Phi_{U_1,\ldots,U_m})$



Barren plateaus

- **Variational quantum algorithms** encode the problem into $\min_{U_1,\ldots,U_m} \mathrm{Cost}(\Phi_{U_1,\ldots,U_m})$

- **Barren plateaus (informal)** iff for most choices of $U_1, \ldots, U_m$ the cost landscape is 'flat' (gradient norm very small).

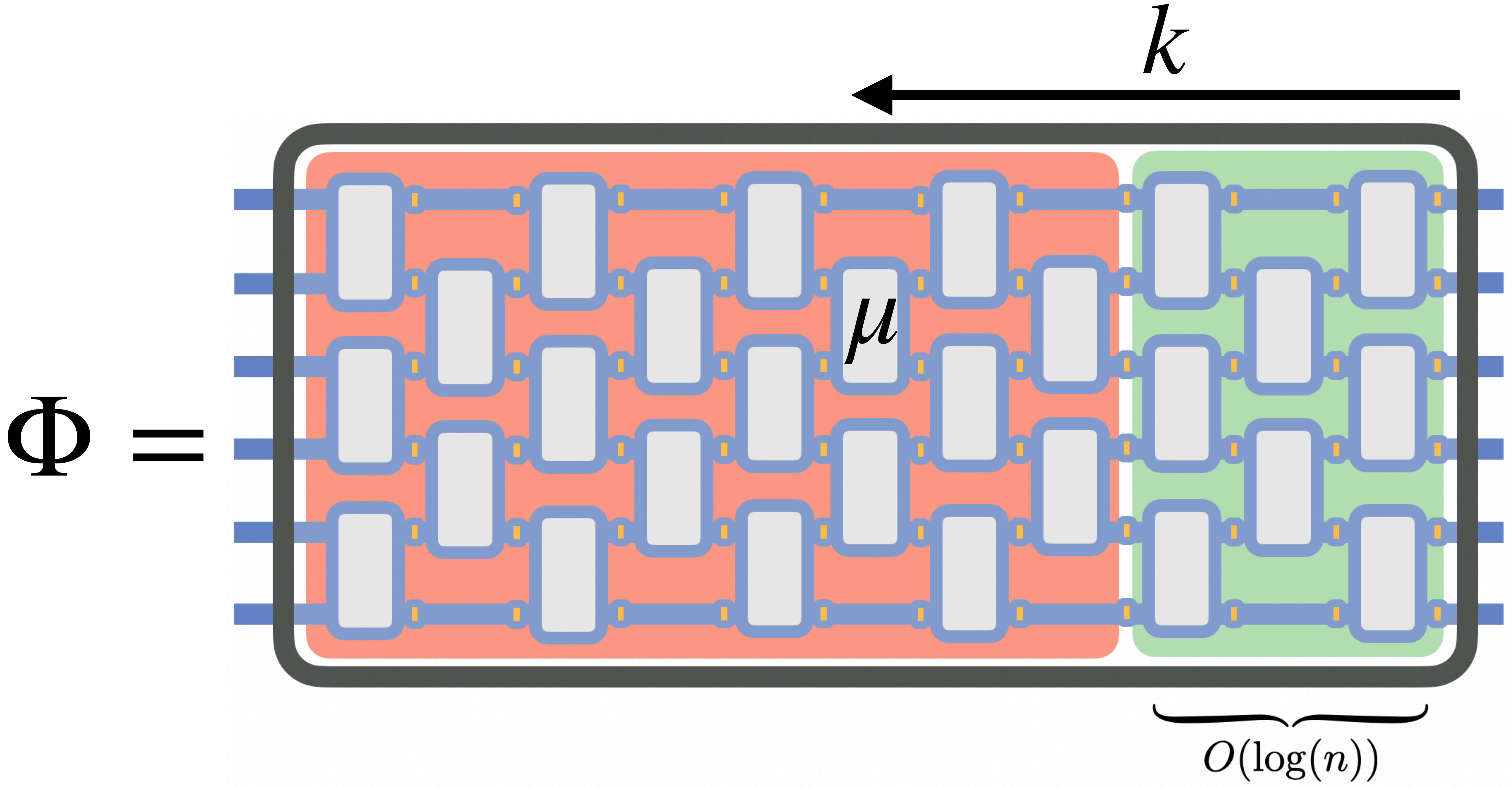
$\Phi =$ 

- Cost functions are made by Pauli expectation values: $C := \mathrm{Tr}(P\Phi(\rho_0))$

- Do variations in the gate μ influence C? Relevant quantity is $\partial_\mu C$.

If $\mathrm{Var}[\partial_\mu C] \leq \exp(-\Theta(n))$, then the influence is very unlikely.

(a.k.a., the gate is **not trainable**)



Cost function:

$$C := \mathrm{Tr}(P\Phi(\rho_0))$$

- **Noiseless** [1]**:** Barren plateaus for depth $\geq \Omega(n)$.

$$\mathrm{Var}[C] \leq \exp(-\Theta(n)),$$

(All the gates are not-trainable) $$\mathrm{Var}[\partial_\mu C] \leq \exp(-\Theta(n))\,.$$

- **Depolarizing noise** [2]**:** Same of before.

- **Non-unital noise [This work]:** No Barren plateaus at any depth for local cost functions.

$$\mathrm{Var}[C] = \Omega(1),$$

$$\mathrm{Var}[\partial_\mu C] = \exp(-\Theta(k))\,.$$

<u>Only</u> the last $\Theta(\log(n))$ layers are trainable!

[1] Barren plateaus in quantum neural network training landscapes. McClean *et al.* Nature Comm. (2018).
[2] Noise-induced barren plateaus in variational quantum algorithms. Wang *et al,* Nature Comm. (2021).

<u>**Compare with**</u>: **[3] Beyond unital noise in variational quantum algorithms: noise-induced barren plateaus and fixed points**. Singkanipa *et al.*, ArXiv. (2024).

(Lack of) Cost concentration

Theorem.

Let $H := \sum_{P \in \{I,X,Y,Z\}^{\otimes n} \setminus I^{\otimes n}} a_P P$ be an Hamiltonian, ρ_0 an initial state, for any depth of the non-unital noisy circuit Φ, we have:

$$\mathbb{E}_\Phi[\mathrm{Tr}(H\Phi(\rho_0))] = 0,$$
$$\mathrm{Var}_\Phi[\mathrm{Tr}(H\Phi(\rho_0))] = \sum_{P \in \{I,X,Y,Z\}^{\otimes n} \setminus I^{\otimes n}} a_P^2 \exp(-\Theta(|P|)),$$

where $|P|$ is the Pauli weight.

Corollaries: - Local cost functions do not suffer from exponential concentration.

(For example, $\mathrm{Var}_\Phi[\mathrm{Tr}(Z_1\Phi(\rho_0))] = \Omega(1)$.)

- Global cost functions suffer from exponential concentration.

(For example, $\mathrm{Var}_\Phi[\mathrm{Tr}(Z^{\otimes n}\Phi(\rho_0))] = \exp(-\Theta(n))$.)

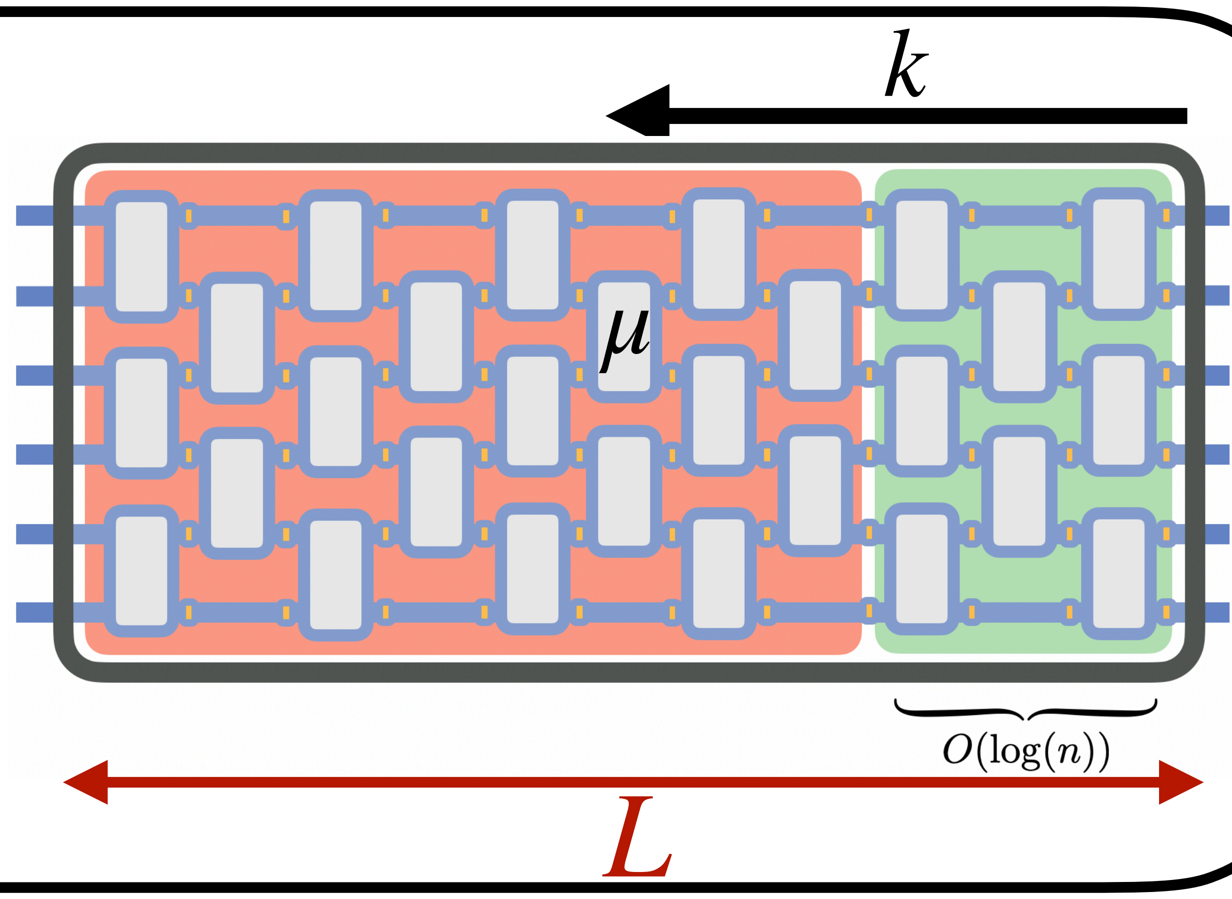
Partial derivatives

Theorem.

Let $C = \mathrm{Tr}(\Phi(\rho_0)P)$, where P is a Pauli, at any depth and noise:

$$\mathbb{E}_\Phi[\partial_\mu C] = 0$$

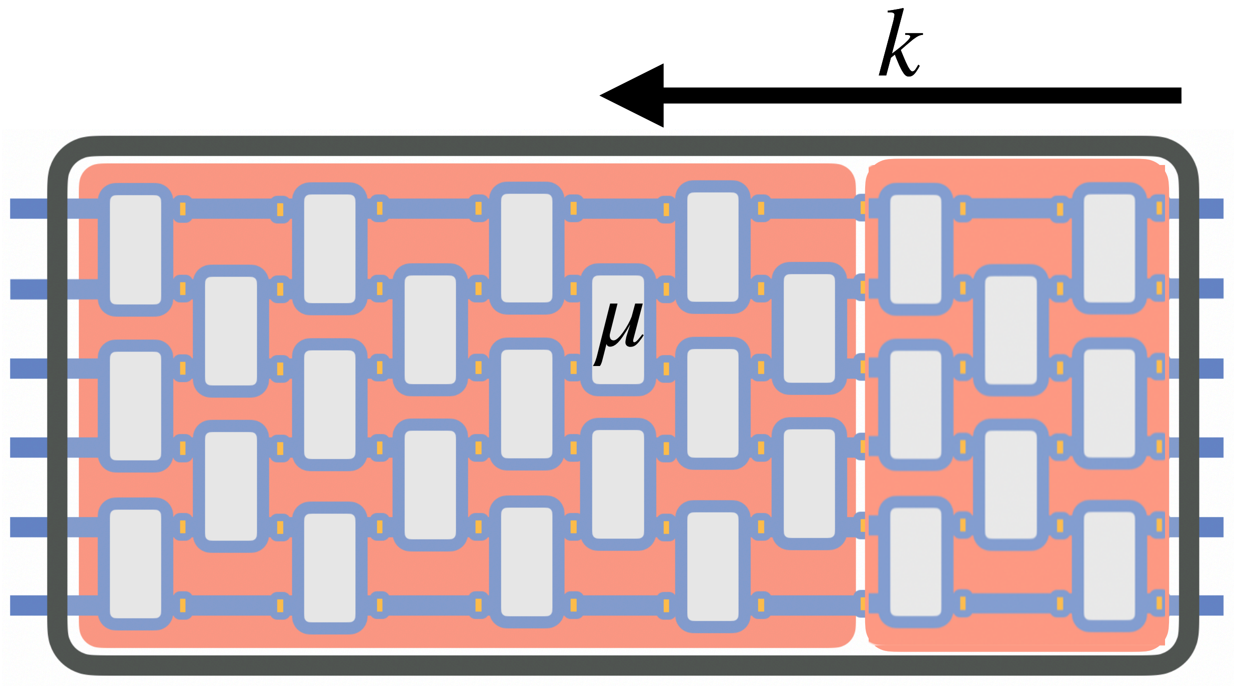
$$\mathrm{Var}_\Phi[\partial_\mu C] \leq \exp(-\Theta(k + |P|)),$$

$\Phi =$ 

- If P is local, then for any non-unital noise $\mathrm{Var}_\Phi[\partial_\mu C] \geq \exp(-\Theta(k))$.

  Corollary: $\mathbb{E}_\Phi[\|\nabla C\|_2^2] = \Omega(1)$.

- If P is global, then for any gate $\mathrm{Var}_\Phi[\partial_\mu C] = \exp(-\Theta(n))$.

$\Phi =$ 

- For the particular case of unital noise, our tools imply: $\mathrm{Var}_\Phi[\partial_\mu C] \leq \exp(-\Theta(L + |P|))$, which improves upon [1].

[1] Noise-induced barren plateaus in variational quantum algorithms. Wang *et al,* Nature Comm. (2021).

Proof ingredients

- Work in the Heisenberg picture. $\mathrm{Tr}(P\Phi(\rho_0)) = \mathrm{Tr}(\Phi^*(P)\rho_0)$

- 2-design properties (of the 2-qubits gates).

- ‘Normal form’ of noisy channels. Any 1-qubit channel is unitarily equivalent to a channel parametrized by:

$$(D_X, D_Y, D_Z), (t_X, t_Y, t_Z) \in \mathbb{R}^3$$

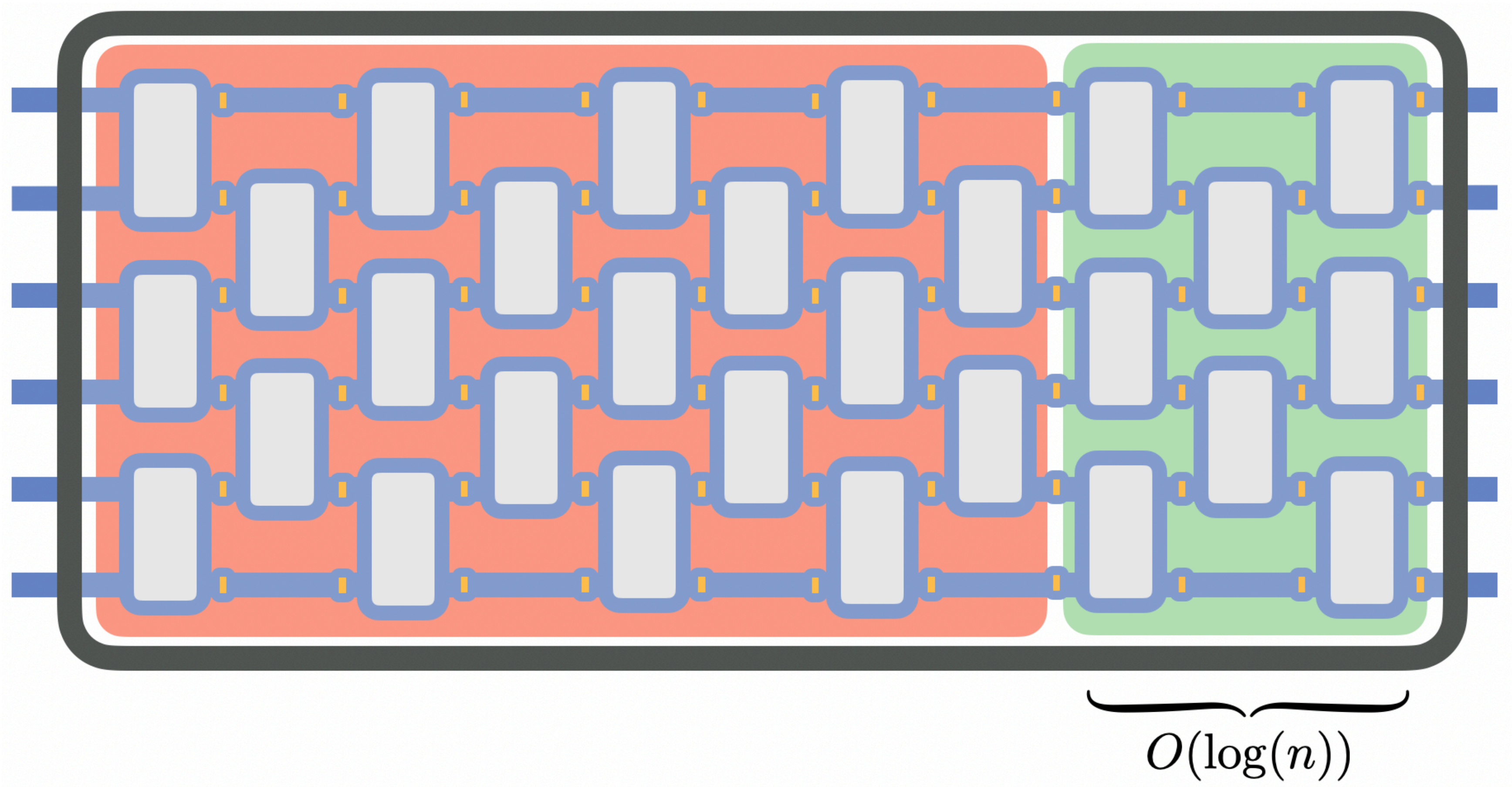
$$\mathcal{N}(I) = I + t_X X + t_Y Y + t_Z Z$$

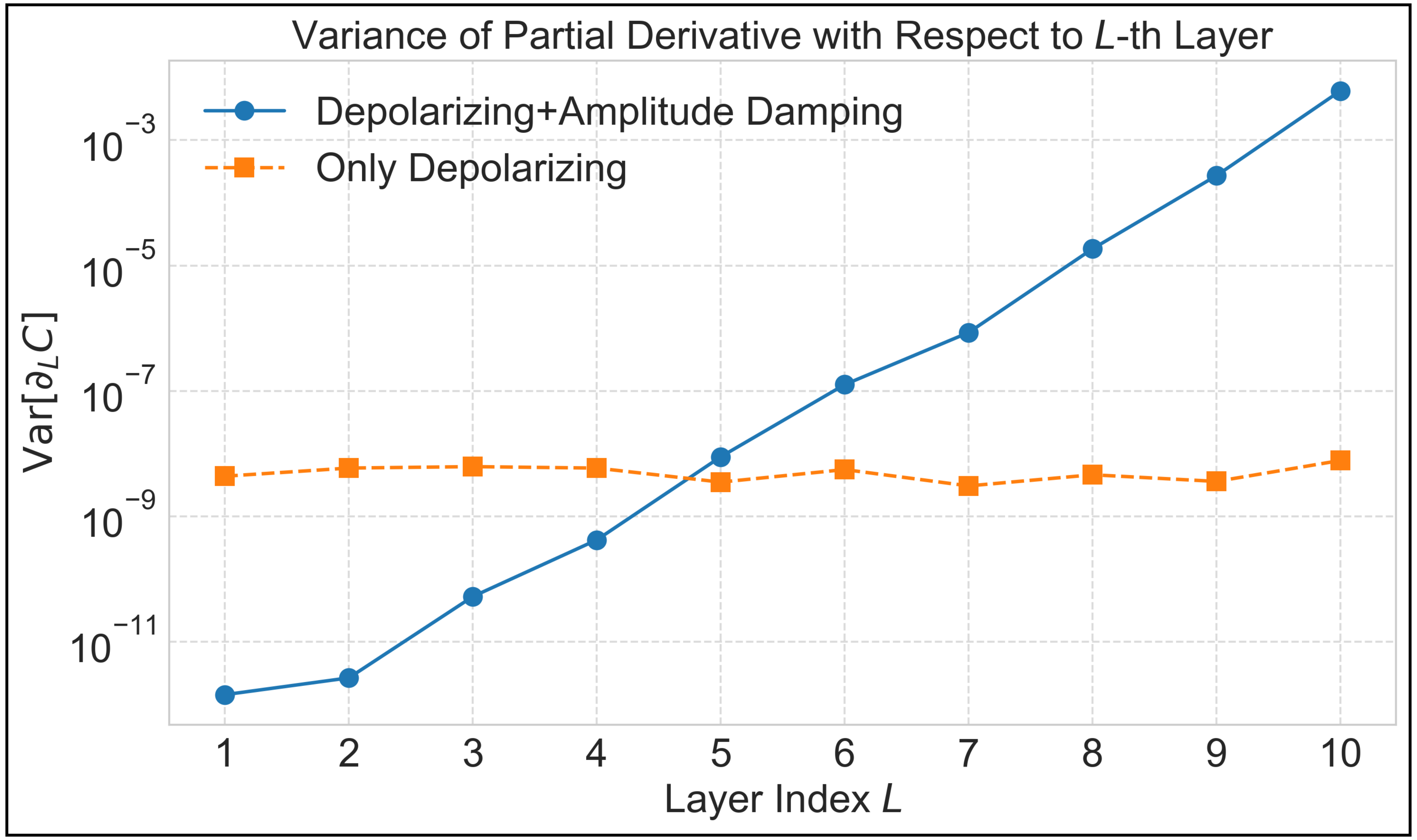
$$\mathcal{N}(X) = D_X X$$

$$\mathcal{N}(Y) = D_Y Y$$

$$\mathcal{N}(Z) = D_Z Z$$

Numerical experiments confirm our results for more structured circuits.





Conclusions

- Any amount of **non-unital noise induces lack of Barren plateaus**, but it ‘truncates’ most circuits to **effectively log-depth**, allowing **efficient classical simulation** in estimating Pauli expectation values.
- **Unless** we carefully **engineer the circuits to take advantage of the noise**, it is **unlikely** that **non-unital** noisy circuits are **preferable over depolarizing** noisy circuits (which are restricted to log depth).
- It is unlikely that noisy quantum circuits provide any quantum advantage.
- Assuming overly simplistic noise models could be misleading.

Open questions

- Complexity of classical simulation of random quantum circuit sampling with non-unital noise [1].

(The depolarizing case was addressed in [2])

Thanks a lot for your attention!

[1] Effect of non-unital noise on random circuit sampling, Fefferman *et al*, QIP (2023).
[2] A polynomial-time classical algorithm for noisy random circuit sampling, Aharonov *et al.*, STOC (2023)