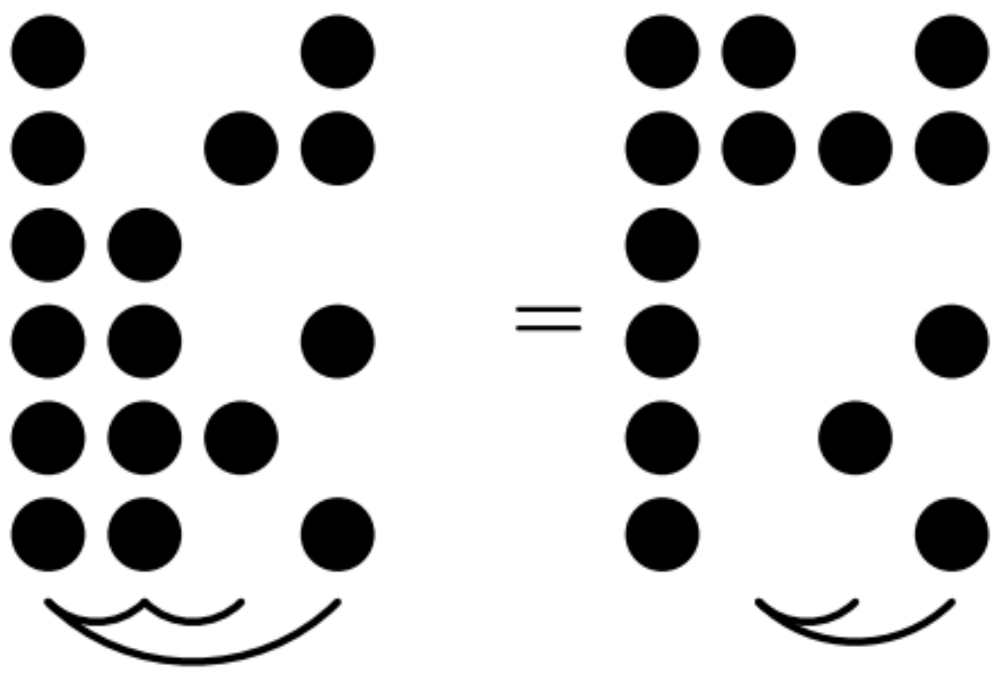
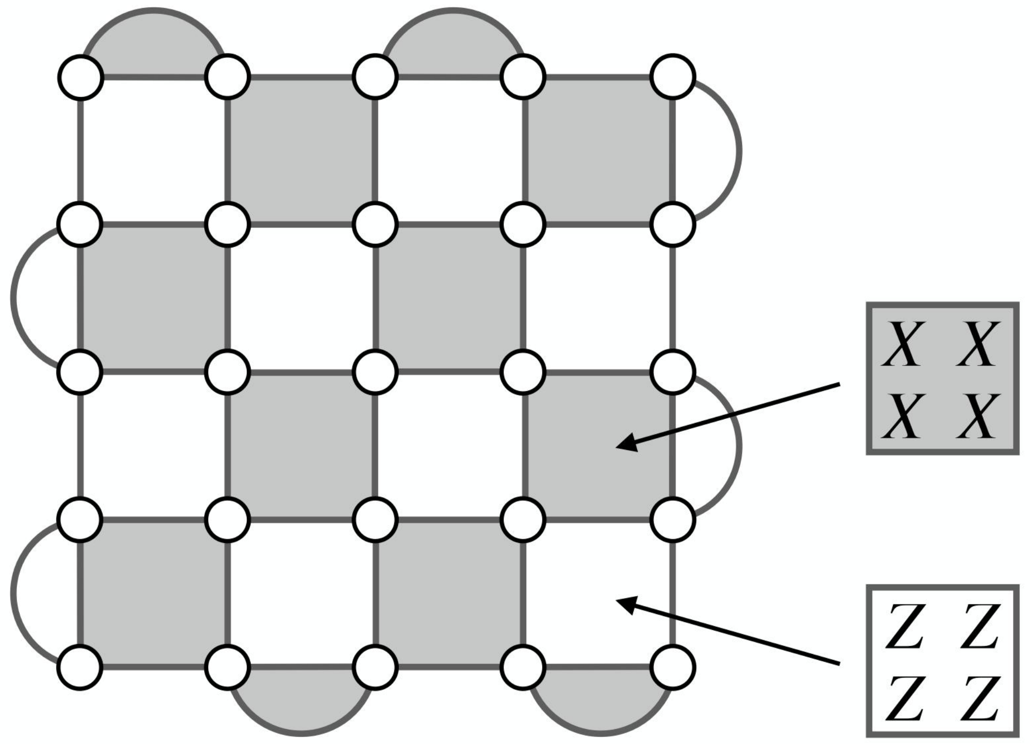
A complete theory of the Clifford Commutant



Antonio Anna Mele
Freie Universität Berlin



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Quantum Physics

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A complete theory of the Clifford commutant

Lennart Bittel, Jens Eisert, Lorenzo Leone, Antonio A. Mele, Salvatore F.E. Oliviero

Joint work with great collaborators:

Lennart Bittel

FREIE UNIVERSITÄT BERLIN

Jens Eisert

FREIE UNIVERSITÄT BERLIN

Lorenzo Leone

FREIE UNIVERSITÄT BERLIN

Salvatore F.E. Oliviero

FREIE UNIVERSITÄT BERLIN

SCUOLA NORMALE SUPERIORE

Outline

- Preliminaries
- Related works
- Overview of our results
    - Generators of the Clifford commutant
    - An easy-to-manipulate basis
    - Orthonormal basis and Dimension of the Clifford commutant
    - Applications

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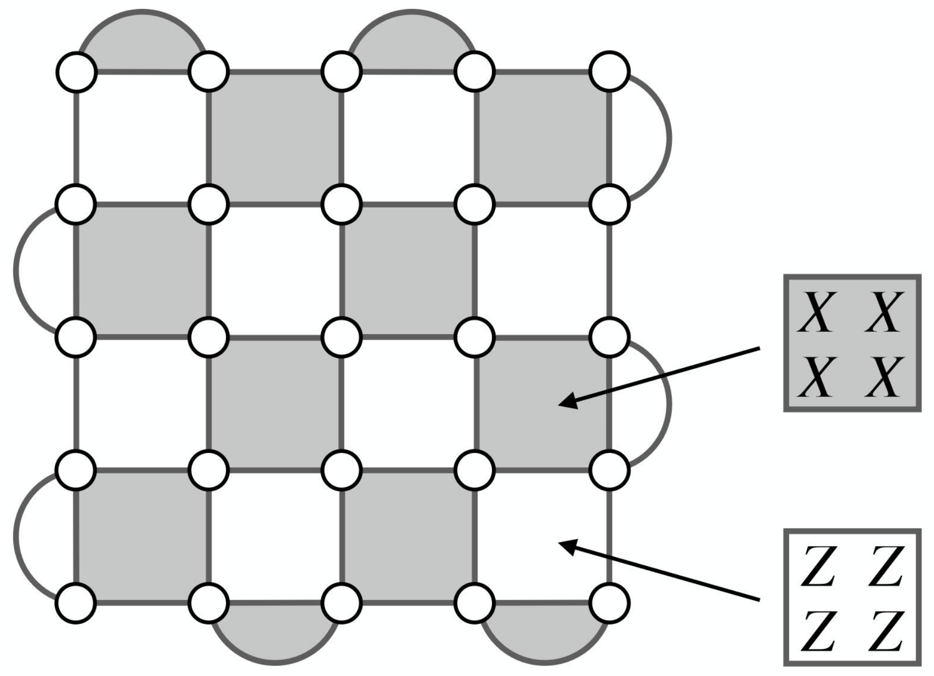
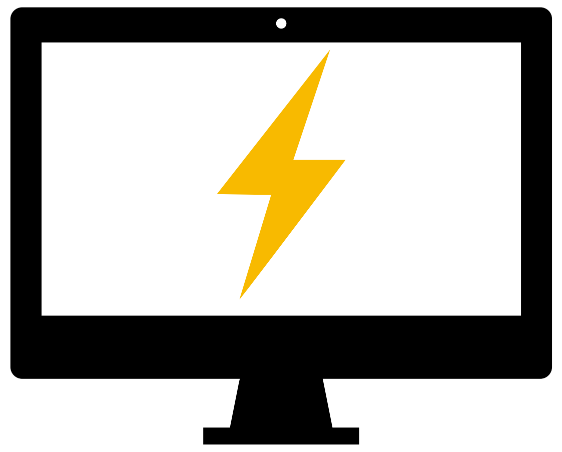
Introduction

The Clifford group is central in quantum information:

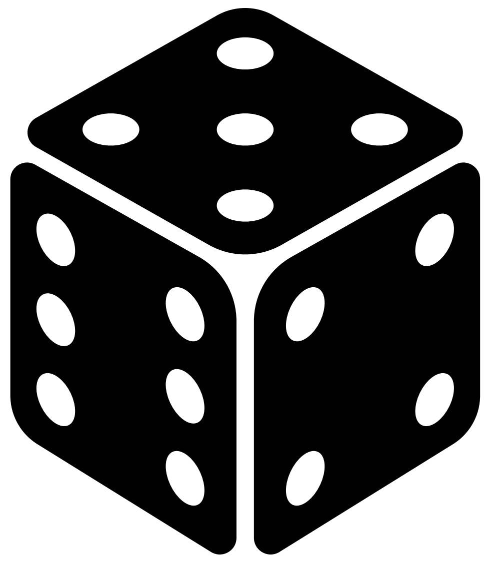


- Quantum Error Correction – stabilizer codes, fault-tolerance.

- Classical simulation – Gottesman–Knill theorem.



- Random Clifford circuits = “nice randomness” (3-designs, benchmarking, tomography)



Preliminaries

- Pauli basis:

$$\mathbb{P}_n := \{I, X, Y, Z\}^{\otimes n}$$

- Clifford group:

$$\mathrm{Cl}_n := \left\{ C \in \mathrm{U}(2^n) \ : \ CPC^\dagger \in \mathbb{P}_n, \ \forall P \in \mathbb{P}_n \right\}.$$

- Commutant of the Clifford group: $d := 2^n, \qquad \mathcal{H} = \mathbb{C}^d.$

$$\mathrm{Com}(\mathrm{Cl}_n, k) \ = \ \left\{ O \in \mathcal{L}(\mathcal{H}^{\otimes k}) \ : \ C^{\dagger \otimes k} O \, C^{\otimes k} = O, \ \forall C \in \mathrm{Cl}_n \right\}.$$

Our work provides a *complete* characterization of such set.

Preliminaries

- The twirling channel over a set $G \subseteq \mathrm{U}(2^n)$ (such as $G = \mathrm{Cl}_n$) is defined as:

$$\Phi_G^{(k)}(\cdot) = \frac{1}{|G|} \sum_{C \in G} C^{\otimes k} (\cdot) C^{\dagger \otimes k}.$$

- Twirling over a group = projection onto its commutant

$$\mathrm{Com}(G, k) \;=\; \left\{ O \in \mathcal{L}(\mathcal{H}^{\otimes k}) \;:\; U^{\dagger \otimes k}\, O\, U^{\otimes k} = O,\ \forall\, U \in G \right\}.$$

- Useful for computing average quantities, and rigorous guarantees in randomized protocols.

Unitary group commutant

- The commutant of the unitary group is given by:

$$\mathrm{Com}(\mathrm{U}(2^n), k) = \mathrm{Span}(V_\pi : \pi \in S_k)$$

Permutation operators **Permutation group**

- Permutation operators:

$$V_\pi = \sum_{i_1,\ldots,i_k=1}^{2^n} |i_{\pi^{-1}(1)}, \ldots, i_{\pi^{-1}(k)}\rangle\langle i_1, \cdots, i_k|.$$

- A distribution over G is a k-design iff:

$$\Phi_G^{(k)}(\cdot) = \Phi_{\mathrm{U(2^n)}}^{(k)}(\cdot)$$

- If G is a subgroup of $\mathrm{U}(2^n)$, then G is a k-design if and only if:

$$\mathrm{Com}(G, k) = \mathrm{Com}(\mathrm{U}(2^n), k)$$

Equivalently, G is a k-design iff the commutant **dimension** of G and $\mathrm{U}(2^n)$ is the same!

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Related works

- Clifford group = exact **3-design** [1] Multiqubit Clifford groups are unitary 3-designs, Zhu, 2015.
- But it fails 'gracefully' to be a 4-design [2] The Clifford group fails gracefully to be a unitary 4-design, Zhu et al, 2015.

$$\Pi_4 := \sum_{P \in \mathbb{P}_n} P^{\otimes 4}$$

$$\Pi_4 \in \mathrm{Com}(\mathrm{Cl}_n, k = 4)$$

$$\Pi_4 \notin \mathrm{Com}(U(2^n), k)$$

(Even if not a 4-design, it is still approximate <u>state</u> 4-design!)

- $(k = 4)$-th Clifford commutant is generated by permutations and Π_4.

$$\mathrm{Com}(\mathrm{Cl}_n, 4) = \mathrm{Span}(\langle V_\pi, \Pi_4 \rangle_{\pi \in S_k})$$

- The k-th Clifford commutant was characterized for $k \leq n - 1$ in:

[Submitted on 22 Dec 2017 (v1), last revised 16 Jan 2021 (this version, v3)]

Schur–Weyl Duality for the Clifford Group with Applications: Property Testing, a Robust Hudson Theorem, and de Finetti Representations

David Gross, Sepehr Nezami, Michael Walter

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Our results (Spoiler/Quick overview)

1) **Generators:**

- The k-th Clifford commutant is generated by

  - Permutations V_π ,

  - Pauli sums

$$\sum_{P \in \mathbb{P}_n} P^{\otimes q}, \quad q \in 2\mathbb{N},\ q \leq k$$

- It suffices to take:

$q \in \{4,6\}$, and if $k \in 4\mathbb{N}$, also $q = k$.

2) **Basis:**

- Products of these Pauli sums yield: the **Pauli monomials**,
- They form a basis for the commutant, easy to manipulate via a **graphical calculus**.

3) **Dimension:**

- Explicit formula for $\dim(\mathrm{Com}(\mathrm{Cl}_n, k))$ for all $n, k \in \mathbb{N}$.
- Computed by: - constructing an **orthonormal basis** (via twirling the Pauli basis),
  - and **counting** its elements.

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Commutant generators

- **Unitary commutant**: span of permutations, generated by SWAPs:

$$\mathrm{Com}(\mathrm{U}(2^n),k)=\mathrm{Span}(V_\pi : \pi\in S_k)\ =\mathrm{Span}\big(\langle \mathrm{SWAP}_{i,j}\rangle_{i,j\in[k]}\big)\,. \qquad \mathrm{SWAP}=\frac{1}{d}\sum_{P\in\mathbb{P}_n}P^{\otimes 2}$$

↑ Permutation operators

- What about the **Clifford commutant**?
  - Consider: $\sum_{P\in\mathbb{P}_n} P^{\otimes v},\quad v\in\{0,1\}^k,\ |v|\equiv 0 \pmod 2$
  - They are in the commutant: $C\left(\sum_{P\in\mathbb{P}_n}P^{\otimes v}\right)C^\dagger=\sum_{P\in\mathbb{P}_n}(CPC^\dagger)^{\otimes v}=\sum_{Q\in\mathbb{P}_n}Q^{\otimes v}$
  - We also show that they are **sufficient to generate it**:

$$\mathrm{Com}(\mathrm{Cl}_n,k)=\mathrm{Span}\left(\left\langle \sum_{P\in\mathbb{P}_n}P^{\otimes v}\right\rangle_{|v|\equiv 0 \pmod 2}\right)$$
$$=\mathrm{Span}\left(\left\langle \sum_{P\in\mathbb{P}_n}P^{\otimes v}\right\rangle_{|v|\in\{2,4,6\}\ \text{ or } |v|=k \text{ if } k\in 4\mathbb{N}}\right).$$

- **Graceful generalization:** For $k=4$, this reduces to Zhu et al. (2015):

$$\mathrm{Com}(\mathrm{Cl}_n,4)=\mathrm{Span}\left(\left\langle \sum_{P\in\mathbb{P}_n}P^{\otimes v}\right\rangle_{|v|\in\{2,4\}}\right).$$

Commutant generators

- So the Clifford commutant is generated by:

$$\mathrm{Com}(\mathrm{Cl}_n, k) = \mathrm{Span}\left(\left\langle \sum_{P\in\mathbb{P}_n} P^{\otimes v} \right\rangle_{|v|\equiv 0 \pmod 2}\right) = \mathrm{Span}\left(\left\langle \sum_{P\in\mathbb{P}_n} P^{\otimes v} \right\rangle_{|v|\in\{2,4,6\} \text{ or } |v|=k \text{ if } k\in 4\mathbb{N}}\right).$$

- How do we prove this?

1) We propose an ***ansatz*** for a **commutant basis**, and prove it is indeed a basis.

2) We show that **every basis element factors** into products of Pauli sums ("**primitives**")

$$\sum_{P\in\mathbb{P}_n} P^{\otimes v}, \quad v\in\{0,1\}^k,\ |v|\equiv 0 \pmod 2)$$

3) Finally, every such Pauli sum factors into products of (acting on **different tensor registers)**

$$\sum_{P\in\mathbb{P}_n} P^{\otimes 2}, \quad \sum_{P\in\mathbb{P}_n} P^{\otimes 4}, \quad \sum_{P\in\mathbb{P}_n} P^{\otimes 6}, \quad \text{and if } k\in 4\mathbb{N}, \ \sum_{P\in\mathbb{P}_n} P^{\otimes k}.$$

- **Question**:

👉 What is this ***ansatz*** for the commutant basis?

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Pauli "monomials": our basis ansatz

- We define a family of operators $\{\Omega(V,M)\}$:
  - $V \in \mathbb{F}_2^{k\times m}$ with **even columns** $(|v_j| \equiv 0 \pmod 2)$, with $m \leq k-1$
  - $M \in \mathrm{Sym}_0(\mathbb{F}_2^{m\times m})$: symmetric, zero diagonal.

Columns of V

$$\Omega(V,M) = \frac{1}{d^m} \sum_{P_1,\ldots,P_m \in \mathbb{P}_n} \left(\prod_{1\leq i<j\leq m} \chi(P_i,P_j)^{M_{ij}} \right) \prod_{j=1}^{m} P_j^{\otimes v_j}.$$

where: $d := 2^n, \quad \chi(P_i,P_j) = \begin{cases} +1, & \text{if } P_i, P_j \text{ commute} \\ -1, & \text{if they anticommute} \end{cases}$

- Pauli monomials lie in the commutant:

$$C^{\otimes k}\,\Omega(V,M)\,(C^\dagger)^{\otimes k} = \Omega(V,M), \qquad \forall\, C \in \mathrm{Cl}_n.$$

(Because Cliffords **map Paulis to Paulis** and **preserve commutation** relations)

Pauli monomials form a basis

$$\mathcal{P} = \{\Omega(V, M) \,|\, V \in \mathrm{Even}(\mathbb{F}_2^{m\times m}) \;:\; \mathrm{rank}(V) = m \,,\; M \in \mathrm{Sym}_0(\mathbb{F}_2^{m\times m}) \,,\; m \in [k-1]\} \,.$$

Matrices with even columns — Symmetric, zero diagonal

Theorem: The set of Pauli monomials $\mathcal{P}$ spans the Clifford commutant:

$$\mathrm{Com}(\mathrm{Cl}_n, k) = \mathrm{Span}(\mathcal{P})$$

Properties:

- For $n \geq k-1$, the operators in $\mathcal{P}$ are linearly independent.
- The number of Pauli monomials is

$$|\mathcal{P}| = \prod_{i=0}^{k-2} (2^i + 1) = 2^{\Theta(k^2)}$$

- **Sanity check:** for $k = 3$, the Clifford group is a 3-design.

Properties of Pauli monomials

- Just **like permutations**:

  - they are "approximately" orthogonal:

$$\mathrm{Tr}(\Omega^\dagger \Omega') = \begin{cases} d^k, & \Omega = \Omega', \\ \leq d^{k-1}, & \text{otherwise.} \end{cases}$$

  - they factorize on qubits:

$$\Omega(V, M) \;=\; \big(\omega(V, M)\big)^{\otimes n}$$

(where ω is a Pauli monomial with $d = 2$)

- **Gauge freedom for $\Omega(V, M)$:**

$$\forall\, A \in \mathrm{GL}(\mathbb{F}_2^{m\times m}) \;\; \exists\, M'_A : \quad \Omega(V, M) \;=\; \Omega(VA, M'_A).$$

That is, arbitrary Gaussian operations on V can be implemented by adjusting M.

- **Normal form:**

$$\exists\, V', V'' : \quad \Omega(V, M) \;=\; \Omega(V', 0)\, \Omega(V'', 0).$$

Normal form → 'phases' (M) can be removed → Ω is just a product of "primitive" Pauli sums.

Here: - V' has all column with hamming weight in $4\mathbb{N}$,

- V'' has all column with hamming weight in $4\mathbb{N} + 2$.

Normal form of Pauli monomials

Projector — Unitary

$$\exists\, V', V'' : \quad \Omega(V, M) \;=\; \Omega(V', 0)\,\Omega(V'', 0).$$

Here:
- V' has all columns with hamming weight in $4\mathbb{N}$,
- V'' has all columns with hamming weight in $4\mathbb{N}+2$.

- $\Omega(V',0)$ is a product of primitives:

$$\frac{1}{d}\sum_{P\in\mathbb{P}_n} P^{\otimes v}, \qquad |v| \in 4\mathbb{N}.$$

These are (proportional to) **projectors**

- Example: $\frac{1}{d}\sum_{P\in\mathbb{P}_n} P^{\otimes 4}$

- $\Omega(V'',0)$ is a product of primitives

$$\frac{1}{d}\sum_{P\in\mathbb{P}_n} P^{\otimes v}, \qquad |v| \in 4\mathbb{N}+2.$$

These are **unitaries**

- Example: $\frac{1}{d}\sum_{P\in\mathbb{P}_n} P^{\otimes 2}$, $\frac{1}{d}\sum_{P\in\mathbb{P}_n} P^{\otimes 6}$

In many applications, the ***unitary*** Pauli monomials are typically **the only ones that *matter***.

(up to exponential corrections, e.g. “stabilizer states are approx. state 4-design”)

Graphical calculus for Pauli monomials

- Many of the previous properties are shown trough a graphical calculus.

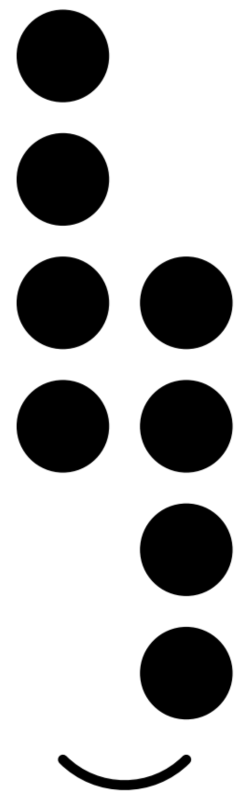
$$\Omega(V, M) = \frac{1}{d^m} \sum_{P_1,\ldots,P_m \in \mathbb{P}_n} \left(\prod_{1 \le i < j \le m} \chi(P_i, P_j)^{M_{ij}} \right) \prod_{j=1}^{m} P_j^{\otimes v_j}.$$

Graphical representation for $\Omega(V, M)$:

Each column v_j of V $\longrightarrow$ a column of black/white dots (1/0)

Each $M_{i,j} = 1$ $\longrightarrow$ a line (a phase) connecting columns i and j

$$\Omega\left(\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) =$$



Algebraic manipulations of Pauli monomials translate into simple diagram moves.

Graphical calculus for Pauli monomials

- Recall the “Gauge freedom”:

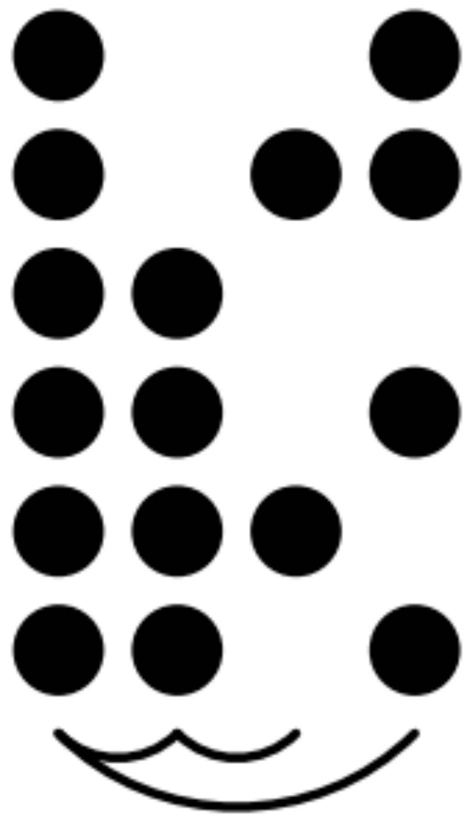
$$\Omega(V, M) = \Omega(VA, M'_A), \qquad A \in \mathrm{GL}(\mathbb{F}_2^m)$$

- Generators of such operations A = nearest-neighbor column addition.

Column addition rules (add col 1 to col 2):

- If $|v_1| \equiv 2 \pmod 4$, add a line between col 1 and col 2.
- **Propagate phases:** any other column connected to col 1 gets a new line to col 2.

$\Omega(V, M) =$



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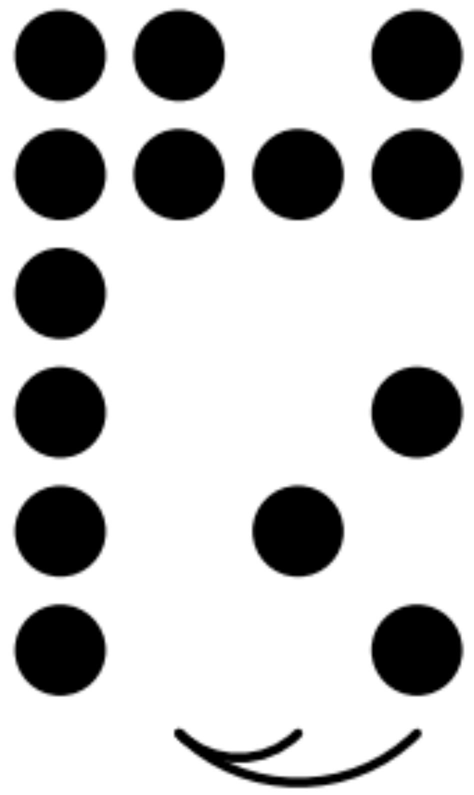

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- These rules allow very fast calculations (e.g. proving the normal form, or magic-measures equivalence).

Pauli monomials form a basis — but why?

Question:

How do we prove that Pauli monomials really form a basis of the commutant?

Idea: Find an invertible map to another operator set already known to be a basis.

Which basis do we already know?

- Image of the Clifford twirling channel = commutant.
- Thus, twirling a full operator basis (e.g. Paulis) ⇒ basis of the commutant!

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Orthonormal basis via Clifford twirling

- Twirling each Pauli basis element ⇒ basis of the commutant!
- Doing this explicitly yields orthogonal operators

$$\mho_I(V,G) = \sum_{\substack{P_1,\ldots,P_m \in \mathbb{P}_n \\ \mathcal{A}(P_1,\ldots,P_m)=G, \\ \text{alg. indep.}}} \prod_{j=1}^{m} P_j^{\otimes v_j}.$$

$V \in \mathrm{Even}(\mathbb{F}_2^{k\times m})$
$G \in \mathrm{Sym}_0(\mathbb{F}_2^{m\times m})$

Adiacency matrix of the anticommutation graph of the Paulis

Sum only over the Paulis which are "algebraic" indipendent
(i.e., they cannot written as product of the others)

Fact:

- $\mho_I(V,G)$ is nonzero iff $\mathrm{rank}(G) \geq 2(m-n)$.
- For all valid V, G, these operators form an orthogonal basis.
- Counting them gives the exact dimension of the commutant.

Dimension of the Clifford commutant

- So the commutant dimension is given by the number of allowed V and G

$$\dim\big(\mathrm{Com}(\mathrm{Cl}_n, k)\big) = \sum_{m=0}^{k-1} \Big(\# \ \ m\text{-dimensional even subspaces of } \mathbb{F}_2^k\Big) \ \times \ \Big(\# \ \ \text{allowed } m \times m \text{ graphs } G\Big).$$

(Exact closed formula is given in the paper.)

- **Asymptotics** (up to constant factors):

$$\dim\big(\mathrm{Com}(\mathrm{Cl}_n, k)\big) \ \simeq \ \begin{cases} 2^{\frac{k^2-3k}{2}}, & 2n \geq k-1, \\ 2^{\,2kn-2n^2-3n}, & 2n < k-1. \end{cases}$$

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Applications (& more)

- The tools developed have several applications, such as:

Magic-state resource theory

(monotones ↔ commutant)

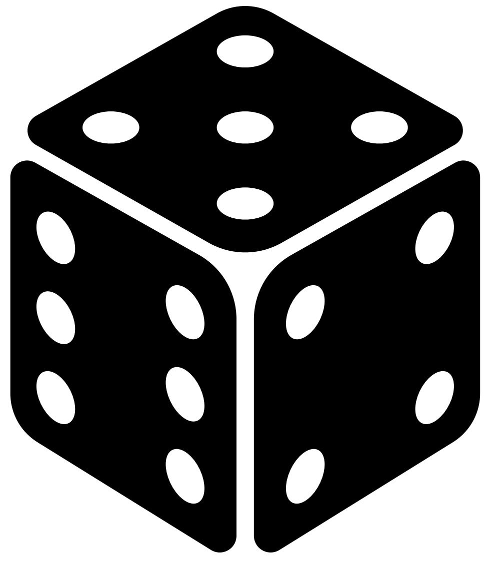
Property testing

(Optimal POVM ↔ commutant)

- Many of the results shown naturally extend to **qudits**.

Summary

- Our main results:
  - Generators of the Clifford commutant: **permutations + 3 primitives**!
  - Easy-to-use basis of **Pauli monomials** (thanks to graphical calculus)
  - **Orthonormal basis** and **Dimension** of the Clifford commutant for all n, k .



Thank you for your attention!