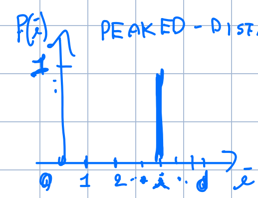
- $X = \{x_1, \dots, x_d\}$ Random Variable $\overset{\text{DEF.}}{\Longleftrightarrow}$ $\begin{cases} \cdot\ p_i \geq 0 \\ \cdot\ \sum_{i=1}^{d} p_i = 1 \end{cases}$

$x_1 \downarrow p_1$, $x_d \downarrow p_d$

Probability to extract x_1

- DEF: (SHANNON ENTROPY of X) $\quad H(X) := -\sum_{i=1}^{d} p_i \log(p_i)$

- $H(X) \geq 0$ $\quad \left(H(X) = 0 \Longleftrightarrow \exists\, j \in \{1, \dots, d\} : p_i = \delta_{i,j} \right)$



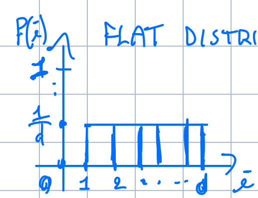
PROOF :

- $0 \leq p_i \leq 1 \Rightarrow -p_i \log(p_i) \geq 0 \Rightarrow H(X) \geq 0$
- $H(X) = 0 \Rightarrow p_i \log(p_i) = 0\ \forall i$, with $p_i \geq 0$ and $\sum_{i=1}^{d} p_i = 1$

$\Rightarrow p_i = 0$ or $\log(p_i) = 0 \Rightarrow \exists\, j : p_i = \delta_{i,j}\ \forall i.$

$\Updownarrow$ $p_i = 1$

- $H(X) \leq \log(d)$ $\quad \left(H(X) = \log(d) \Longleftrightarrow p_i = \frac{1}{d}\ \forall i \right)$



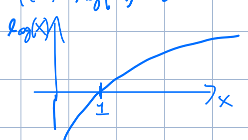
PROOF :

- DEF : $f(x)$ is CONVEX (CONCAVE) $\Longleftrightarrow f\left(\sum_{i=1}^{d} p_i x_i \right) \leq (\geq) \sum_{i=1}^{d} p_i f(x_i)$ where $\cdot\ x_i \in \mathbb{R}$, $\cdot\ p_i \in [0,1]$, $\cdot\ \sum_{i=1}^{d} p_i = 1$

- $H(X) = \sum_{i=1}^{d} p_i \left(-\log(p_i) \right) = \sum_{i=1}^{d} p_i \left(\log\left(\frac{1}{p_i} \right) \right) =$

$\leq \log\left(\sum_{i=1}^{d} p_i \cdot \frac{1}{p_i} \right) = \log(d)$

- $f(x) = \log(x)$ is CONCAVE



- "$\Rightarrow$": If $p_i = \frac{1}{d}\ \forall i \Rightarrow H(X) = \log(d)$
- "$\Leftarrow$": The inequality is saturated if $f\left(\sum_i p_i x_i\right) = \sum_i p_i f(x_i)$ with $x_i = \frac{1}{p_i}$.
  - Now $f(x) = \log(x)$ is strictly CONCAVE $\Rightarrow \sum_i p_i f(x_i) = f\left(\sum_i p_i x_i\right) \Leftrightarrow x_i = \bar{x}\ \forall i$.

  $\Rightarrow \frac{1}{p_i} = \bar{x} \underset{\sum p_i = 1}{\Rightarrow} p_i = \frac{1}{d}\ \forall i$.

- DEF. (JOINT PROB. DISTRIBUTION $P(X,Y)$)
  - $X \equiv \{x_1, \dots, x_d\}$, $Y \equiv \{y_1, \dots, y_D\}$

  $P(x,y)$ joint prob. distribution $\overset{\text{DEF.}}{\Longleftrightarrow}$ $\begin{cases} \cdot\ P(x,y) \geq 0 \quad \forall x \in X, y \in Y \\ \cdot\ \sum_{\substack{x \in X \\ y \in Y}} P(x,y) = 1 \end{cases}$

- DEF. (JOINT ENTROPY $H(X,Y)$)

$$H(X,Y) = -\sum_{\substack{x \in X \\ y \in Y}} P(x,y) \log\left(P(x,y)\right)$$

DEF. (MARGINAL DISTRIBUTION $P(x)$ and $P(y)$ given $P(x,y)$).

- Given JOINT PROB. DISTRIB. $P(x,y)$, we define:

$$P(x) := \sum_{y \in Y} P(x,y) \qquad \text{(marginal distrib. of } X\text{.)}$$

$$P(y) := \sum_{x \in X} P(x,y) \qquad \text{(marginal distrib. of } Y\text{.)}$$

$\left(\begin{array}{l} \cdot\ P(x) \geq 0,\ \sum_x P(x) = 1 \\ \cdot\ H(X) = -\sum_{x \in X} P(x) \log(P(x)) \end{array} \right)$

$\left(\begin{array}{l} \cdot\ P(y) \geq 0,\ \sum_y P(y) = 1 \\ H(Y) = -\sum_{y \in Y} P(y) \log(P(y)) \end{array} \right)$

• TH ① (SUB-ADDITIVE PROPERTY OF $H(X,Y)$)

$$H(X,Y) \leq H(X) + H(Y) \qquad \left(H(X,Y) = H(X) + H(Y) \iff \underset{\text{INDIP. RANDOM VARIABLES}}{P(X,Y) = P(x)\cdot P(y)} \right)$$

PROOF:

$$H(x,y) - H(x) - H(Y) = -\sum_{x,y} P(x,y) \log(P(x,y)) + \sum_{x} P(x) \log(P(x)) + \sum_{y} P(y) \log(P(y))$$

$$= -\sum_{x,y} P(x,y) \log(P(x,y)) + \sum_{x,y} P(x,y) \log(P(x)) + \sum_{x,y} P(x,y) \log(P(y))$$

- $P(x) = \sum_{y} P(x,y)$
- $P(y) = \sum_{x} P(x,y)$

$$= -\sum_{x,y} P(x,y) \log\left(\frac{P(x,y)}{P(x)\,P(y)}\right) = \sum_{x,y} P(x,y) \log\left(\frac{P(x)\,P(y)}{P(x,y)}\right)$$

$$\leq \log\left(\sum_{x,y} \cancel{P(x,y)} \frac{P(x)\,P(y)}{\cancel{P(x,y)}}\right) = \log(1) = 0$$

(• $\log(\cdot)$ CONCAVE)

- $\sum_{x} P(x) = 1$
- $\sum_{y} P(y) = 1$

• Since $\log(\cdot)$ is strictly CONCAVE $\implies$ the inequality is saturated when $\frac{P(x)\,P(y)}{P(x,y)} = \text{CONST} \;\; \forall x,y$.

$\iff P(x,y) = P(x)\cdot P(y)$.

Product distribution.

TH ②

$$H(X) \leq H(X,Y) \quad \left(\text{"="} \iff P(x,y) = P(x)\,\delta_{y,\bar{y}_0} \right)$$

$$H(Y) \leq H(X,Y) \quad \left(\text{"="} \iff P(x,y) = P(y)\,\delta_{x,\bar{x}_0} \right)$$

$$\left(\text{or} \quad \max\left(H(X),H(Y)\right) \leq H(X,Y) \right)$$

PROOF:

$$H(X) - H(X,Y) = -\sum_x \underbrace{P(x)}_{\sum_y P(x,y)} \log\left(P(x)\right) + \sum_{x,y} P(x,y) \log\left(P(x,y)\right)$$

$$= -\sum_{x,y} P(x,y) \log\left(\frac{P(x)}{P(x,y)}\right)$$

$$= \sum_{x,y} P(x,y) \log\left(\frac{P(x,y)}{P(x)}\right) \underset{\log(\cdot)\ \text{CONCAVE}}{\leq} \log\left(\sum_{x,y} \frac{P^2(x,y)}{P(x)}\right)$$

$$\leq \log\left(\sum_{x,y} \frac{P^2(x,y)}{P(x)}\right) \leq \log\left(\sum_x \frac{P^2(x)}{P(x)}\right) = \log(1) = 0 \quad \Rightarrow \square$$

- $$\frac{P^2(x,y)}{P(x)} = \frac{P(x,y)}{P(x)} \cdot P(x,y) \leq 1 \cdot P(x,y)$$
  - $P(x) = \sum_{y'} P(x,y') \Rightarrow \frac{P(x,y)}{P(x)} \leq 1$
  - SATURATED $\iff \frac{P(x,y)}{P(x)} = 1 \iff P(x,y) = P(x) \cdot \delta_{y,\bar{y}_0}$
- $\log(\cdot)$ monotone

- We'll see that $\max\left(H(X),H(Y)\right) \leq H(X,Y)$ is violated in the "QUANTUM VERSION" due to ENTANGLEMENT.

DEF: (CONDITIONAL PROBABILITY $P(X|Y)$)

$$P(x|y) := \frac{P(x,y)}{P(y)}$$

(Fixed y, $P(x|y)$ is a prob. dist.: $\begin{cases} \cdot\ P(x|y) \geq 0 \\ \cdot\ \sum_x P(x|y) = 1 \end{cases}$)

$\sum_x P(x,y) = P(y)$

This defines a random variable $X|_y$ distributed according to $P(x|y)$.

$$P(y|x) := \frac{P(x,y)}{P(x)}$$

(Fixed x, $P(y|x)$ is a prob. dist.: $\begin{cases} \cdot\ P(y|x) \geq 0 \\ \cdot\ \sum_x P(y|x) = 1 \end{cases}$)

This defines a random variable $Y|_x$

OBS! We have $\sum_x P(x|y)\,P(y) = P(x)$

DEF: (CONDITIONAL ENTROPY $H(X|Y)$)

$$H(X|Y) := \mathbb{E}_Y\, H(X|_y) = \mathbb{E}_Y\left(-\sum_x P(x|y)\log(P(x|y))\right)$$

COR:

$$H(X|Y) = -\sum_{x,y} P(x,y)\log(P(x|y))$$
$$= H(X,Y) - H(Y)$$

PROOF:

- $H(X|Y) = \mathbb{E}_Y\left(-\sum_x P(x|y)\log(P(x|y))\right)$

$$= -\sum_{x,y} P(x|y)\,P(y)\log(P(x|y)) =$$

$$\underset{P(x|y):=\frac{P(x,y)}{P(y)}}{=} -\sum_{x,y} P(x,y)\log(P(x|y))$$

- $H(X|Y) = -\sum_{x,y} P(x,y)\log(P(x|y)) = -\sum_{x,y} P(x,y)\log\left(\frac{P(x,y)}{P(y)}\right) =$

$$= -\sum_{x,y} P(x,y)\log(P(x,y)) + \sum_{y,x} \underbrace{P(x,y)}_{P(y)}\log(P(y)) = H(X,Y) - H(Y)$$

OBS.) $H(X|Y) \geq 0$

PROOF:

$$H(X|Y) = H(X,Y) - H(Y) \geq 0$$

$\uparrow$ $\boxed{\text{THEOREM } ②}$

(This inequality does not hold for the Quantum version of $H(X|Y)$ because TH ② is violated.)

DEF. (MUTUAL INFORMATION)

$$I(X:Y) := H(X) + H(Y) - H(X,Y)$$

OBS.)

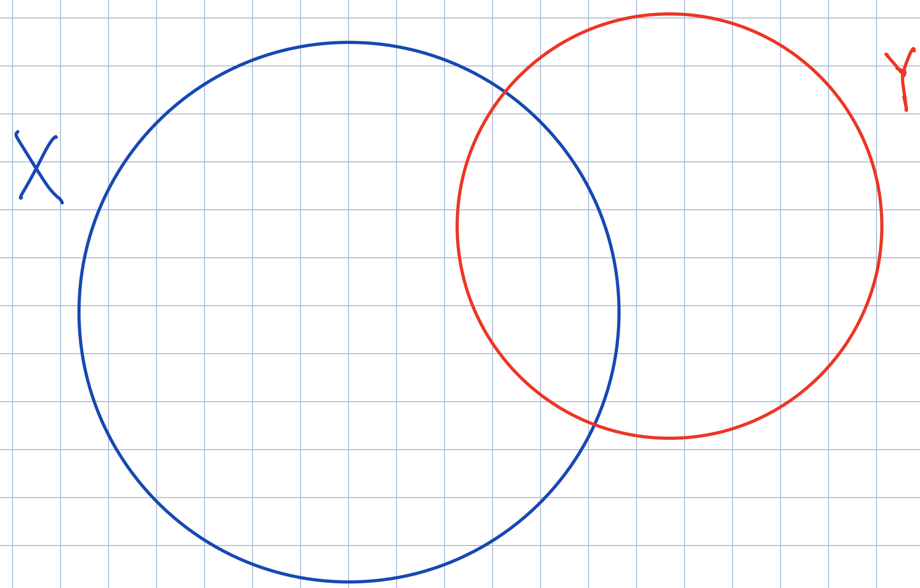
- $I(X:Y) \geq 0$ $\uparrow$ (SUB-ADD.) (TH.②)
- $I(X:Y) = I(Y:X)$ $\uparrow$ $H(X,Y) = H(Y,X)$
- $I(X:Y) = H(X,Y) - H(X|Y) - H(Y|X) = H(Y) - H(Y|X)$

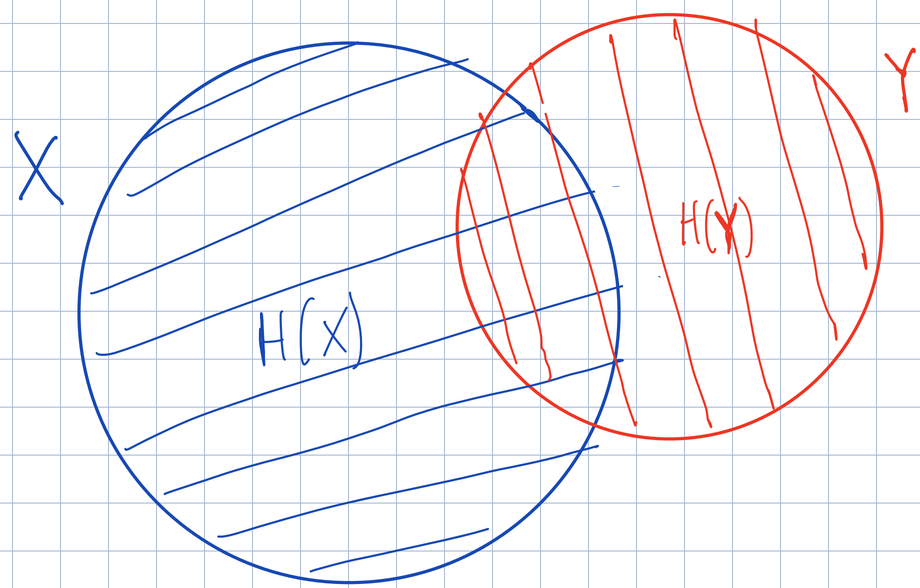
  $I(X:Y) = \big(H(X,Y) - H(X|Y)\big) + \big(H(X,Y) - H(Y|X)\big) - H(X,Y)$

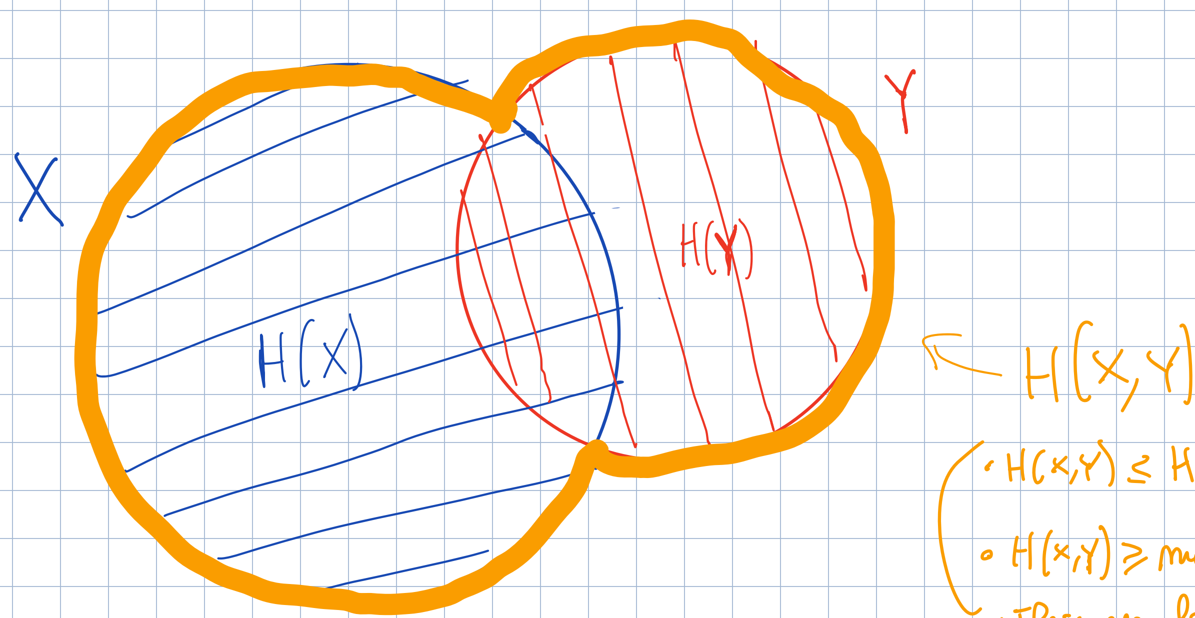
  - $H(X|Y) = H(X,Y) - H(Y)$
  - $H(Y|X) = H(X,Y) - H(X)$
- $I(X:Y) \leq \min\big(H(X), H(Y)\big)$

  $I(X:Y) = H(X) + H(Y) - H(X,Y) \leq H(X) + H(Y) - \max\big(H(X), H(Y)\big) = \min\big(H(X), H(Y)\big)$

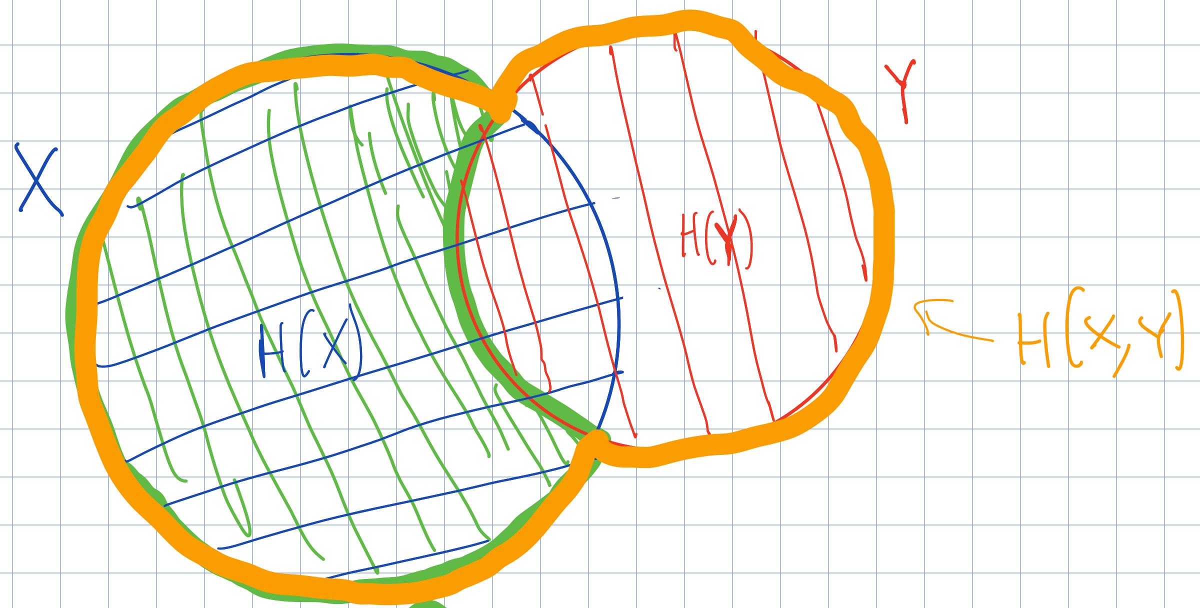
GRAPHICAL INTERPRETATION



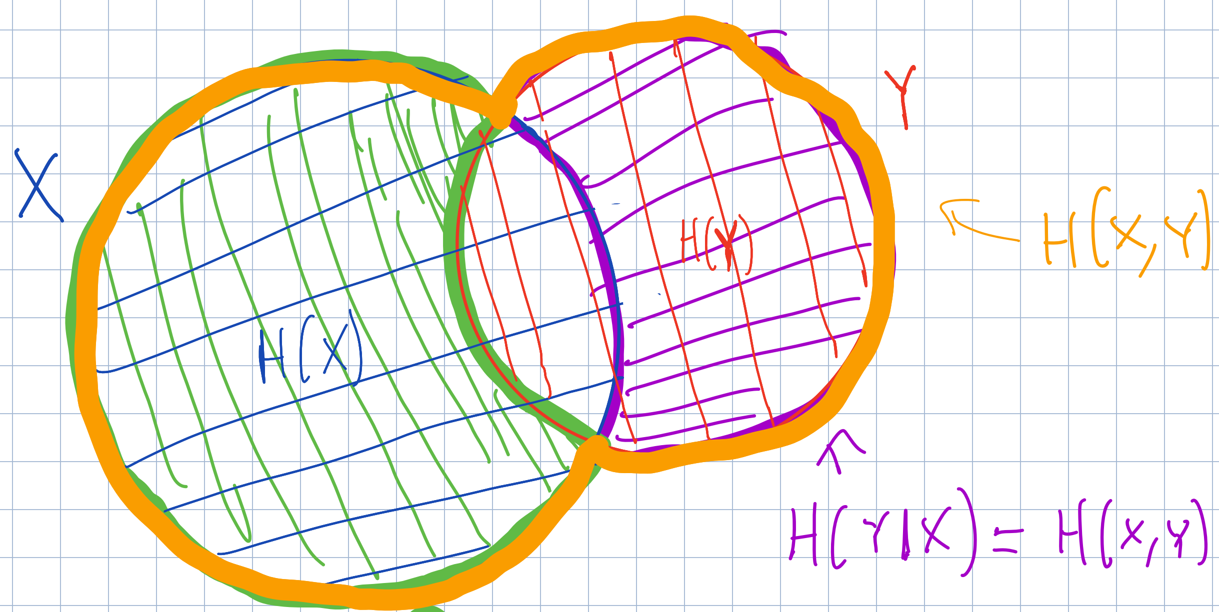




- $H(X,Y) \leq H(X) + H(Y)$
- $H(X,Y) \geq \max(H(X,Y))$
- They are obvious here!

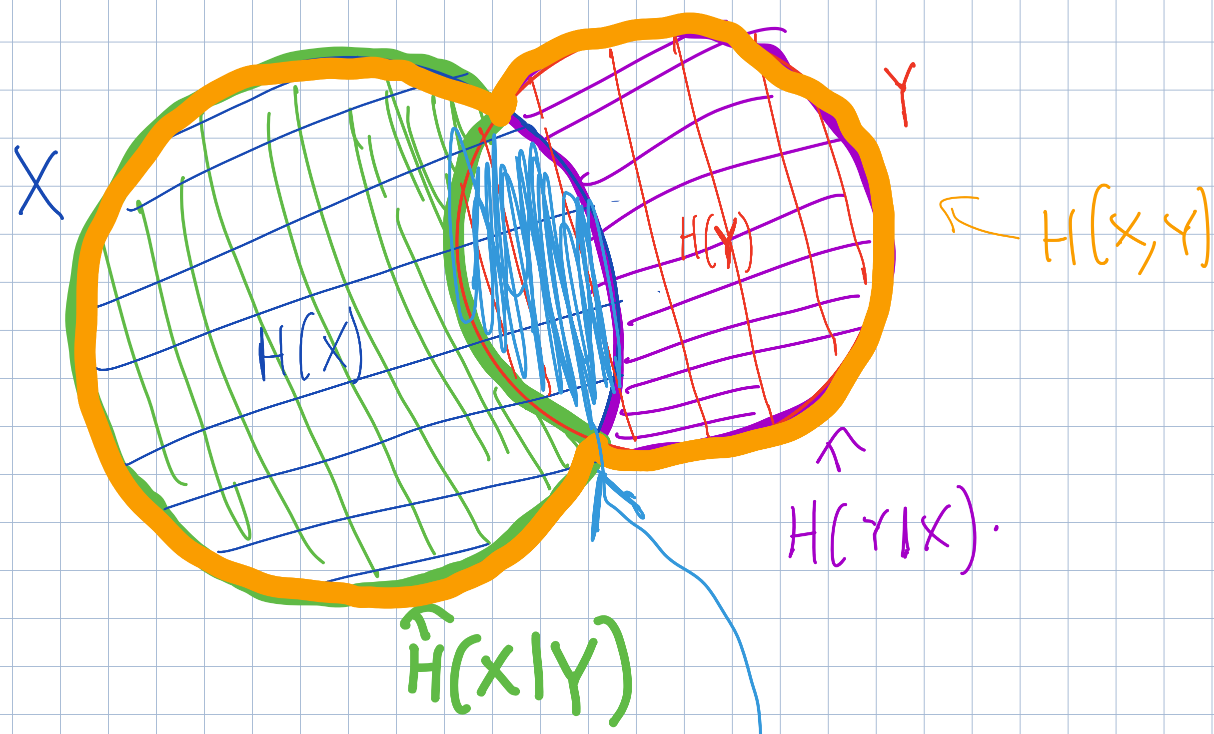


$H(X|Y) = H(X,Y) - H(Y)$



$H(Y|X) = H(X,Y) - H(X)$

$H(X|Y)$



$$I(X:Y) = H(X,Y) - H(X|Y) - H(Y|X)$$
$$= H(X) - H(X|Y)$$
$$= H(Y) - H(Y|X)$$

- DEF: (RELATIVE ENTROPY)

- $p(x) \geq 0$; $\sum_x p(x) = 1$. $q(x) \geq 0$; $\sum_x q(x) = 1$.

$$H\left(p(x) \,\|\, q(x)\right) := \sum_x p(x) \log_2\left(\frac{p(x)}{q(x)}\right) =$$

$$= -H(p) - \sum_x p(x) \log_2\left(q(x)\right)$$

• OBS. $H(p(x) \| q(x)) \geq 0$ $\left(H(p(x) \| q(x)) = 0 \iff p(x) = q(x) \ \forall x \right)$

PROOF:

$$H(p(x) \| q(x)) := \sum_x p(x) \log_2\left(\frac{p(x)}{q(x)}\right) \geq \sum_x \frac{p(x)}{\ln(2)}\left(1 - \frac{q(x)}{p(x)}\right)$$

• $e^x \geq 1 + x$

$\Rightarrow x \geq \ln(1+x)$

$\Rightarrow y - 1 \geq \ln(y) = \log_2(y) \ln(2)$

$\Rightarrow -\log_2(y) \geq \dfrac{1-y}{\ln(2)}$

• $\log_2\left(\frac{p}{q}\right) = -\log_2\left(\frac{q}{p}\right)$

$$= \frac{1}{\ln(2)}(1 - 1) = 0$$

• The inequality $\log_2(y) \geq \frac{1}{\ln(2)}(1-y)$ is saturated $\iff y = 1 \iff p(x) = q(x)$

OBS:

• $$H(p(x,y) \| p(x) \cdot p(y)) \underset{\text{DEF}}{=} -H(X,Y) + H(X) + H(Y) \underset{\text{OBS.}}{\geq} 0$$

$$\Rightarrow H(X,Y) \leq H(X) + H(Y)$$

• $$H(p(x) \| q(x)) \underset{q(x) = \frac{1}{d} \ \forall x}{=} -H(X) - \underbrace{\sum_x p(x)}_{1} \log\left(\frac{1}{d}\right) \underset{\text{OBS}}{\geq} 0 \Rightarrow H(X) \leq \log(d)$$